HONORS EXAM IN REAL ANALYSIS I, II AND COMPLEX ANALYSIS

Answer as many of the following questions or parts of questions as you can. You may quote and use standard results but you need to fully explain your reasons. Show all work and fully support all answers. Good luck.

1. Let $M = (X, \rho)$ be a metric space and let $f: X \to X$ be a function on M. Show that the following three definitions of continuity on M are equivalent.

(a) At each point $x \in X$ the following holds. Given $\epsilon > 0$ there is a $\delta > 0$ such that if $\rho(x, y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$.

(b) If $\mathcal{O}$ is an open set in M, then $f^{-1}(\mathcal{O})$ is open.

(c) At each point $x \in X$ the following holds. If $\{x_n\}$ converges to x then the sequence $\{y_n\} = \{f(x_n)\}$ converges to $y = f(x)$.

2. Let $M = (X, \rho)$ be a metric space and let $f: X \to X$ be a function on M.

(a) Define what it means for f to be *uniformly continuous* on M.

(b) Show that if M is a compact metric space and if f is continuous at each point of X then f is uniformly continuous on M.

(c) Provide an example of a real–valued function continuous at each point of $[0, 1]$ but not uniformly continuous on $(0, 1)$. Fully justify your answer.

3. Let f be a real–valued function defined on an interval $[a, b]$. We say that f is *Lipschitz continuous* on $[a, b]$ provided that there is a constant M such that for every x, $y \in [a, b]$, $|f(x) - f(y)| \leq M|x - y|$. We define the *total variation* of f on $[a, b]$, denoted $T_a^b(f)$ as the supremum of the quantity $\sum_{i=1}^{k} |f(x_{i-1}) - f(x_i)|$ taken over all subdivisions $a = x_0 < x_1 < \cdots < x_k = b$ of the interval $[a, b]$.

(a) Show that if f is Lipschitz continuous on $[a, b]$ then $T_a^b(f) < \infty$.

(b) Show that if $f(0) = 0$ and $f(x) = x^2 \sin\left(\dfrac{1}{x^2}\right)$ if $x \neq 0$, then $T_{-1}^1(f) = \infty$.

4. Given a real–valued sequence $\{a_n\}_{n=1}^{\infty}$, define the sequence $\{\sigma_n\}_{n=1}^{\infty}$, called the *sequence of arithmetic means*, by $\sigma_n = \dfrac{1}{n} \sum_{k=1}^{n} a_k$.

(a) Show that if $\lim a_n = a$ then $\lim \sigma_n = a$.

(b) Show that if $a_n = (-1)^n$, then $\lim a_n$ does not exist while $\lim \sigma_n = 0$.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a real–valued sequence. Show that if every subsequence of $\{a_n\}$ has in turn a subsequence that converges to the real number a, then $\lim a_n = a$.

6. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous, real–valued functions on $\mathbf{R}$ and let f also be a continuous real–valued function on $\mathbf{R}$. Prove or find a counterexample to each of the following statements.

(a) If $f_n \to f$ pointwise on $[0,1]$ then

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx.$$

(b) If $f_n \to f$ uniformly on $[0,1]$ then

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx.$$

(c) If $f_n \to f$ pointwise on $[0,1]$ then

$$\sup_n \int_0^1 |f_n(x)|\,dx < \infty.$$

(d) If $f_n \to f$ uniformly on $[0,\infty)$ then

$$\int_0^\infty f_n(x)\,dx \to \int_0^\infty f(x)\,dx.$$

7. Show that the transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by $u = e^x\cos(y)$, $v = e^x\sin(y)$ is invertible in a neighborhood of each point $(u_0, v_0) \in \mathbf{R}^2$ but is not globally one-to-one.

8. (a) Define what it means for a subset M of $\mathbf{R}^n$ to be a *k–dimensional manifold (in $\mathbf{R}^n$)*.

(b) Prove using the definition you wrote in part (a) that the $(n-1)$–sphere

$$S^{n-1} = \{x \in \mathbf{R}^n\colon |x| = 1\}$$

is an $n-1$ dimensional manifold in $\mathbf{R}^n$.

9. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \frac{2\pi}{3}$.

10. (a) Prove Cauchy's estimate: If $f(z)$ is analytic in $B(a;R)$ (the open ball in $\mathbf{C}$ with center a and radius R), and $|f(z)| \le M$ for all $z \in B(a;R)$ then

$$|f^{(n)}(a)| \le \frac{n!M}{R^n}.$$

(b) Show that an entire function satisfying $|f(z)| \le C|z|^{1/2}$ for all z must be constant.