SWARTHMORE COLLEGE May 1998

# ANALYSIS HONORS EXAM
# REAL ANALYSIS I AND COMPLEX ANALYSIS

Please explicitly state any results you use in answering the following questions.

1. Denote the subset $\{(x, y) : x^2 + y^2 = 1\}$ of the plane by $S^1$. Show that $S^1$ is connected.
2. Is $C([0, 1])$ compact with the metric
$$d(f, g) = \max_{p \in [0,1]} \{|f(p) - g(p)|\}?$$
Why or why not?
3. Prove that the function
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$
is differentiable but that
$$f(x) \neq \int_0^x f'(x) dx.$$
4. Approximate
$$\int_0^{.1} \sin(x^2) dx$$
up to three decimal places.
5. Suppose $C$ is a convex, closed curve in the plane. Namely, if you pick any two points on the curve and join them by a line, the line lies entirely inside the closed curve. (Like an oval.) Assume the curve is smooth.
   Let $P$ be an arbitrary point on the curve. Prove that it is possible to find two other points $Q$ and $R$ on the curve so that $PQR$ is an equilateral triangle.
6. Prove Schwarz' Lemma:
   Suppose that $f$ is analytic in the unit disc, that $|f(z)| \leq 1$ for all $z$ in the unit disc and that $f(0) = 0$. Then
   (a) $|f(z)| \leq |z|$
   (b) $|f'(0)| \leq 1$
   with equality in either of the above if $f(z) = e^{i\theta} z$.
7. Evaluate
$$\int_{|z|=2} z e^{\frac{3}{z}} dz.$$