Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra-A

Spring 2003

Instructions: This exam contains 9 problems. Try to solve *six* problems as completely as possible. Beyond that, turn in any solutions or partial solutions that you can get done.

Advice: I am interested in your thoughts on the problem even if they do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases that you can think of. Where there are multiple parts to a problem, you might be able to answer a later part without solving all the earlier ones.

1. Let $\phi : G \to H$ be a group homomorphism. Let $x, y \in H$ be in the image of ϕ. Find a bijection between the sets $\phi^{-1}(x)$ and $\phi^{-1}(y)$. Justify your answer.

2. (a) Let σ be an automorphism of a group G (an isomorphism from G onto G). Prove that σ permutes the conjugacy classes of G. That is if $\mathcal{K}$ is a conjugacy class of G, then $\sigma(\mathcal{K})$ is a conjugacy class of G. (*Recall that $\mathcal{K}$ is a conjugacy class means that for some $h \in G$, $\mathcal{K} = \{g^{-1}hg | g \in G\}$.*)

   (b) Prove that any automorphism of the symmetric group S_5 sends transpositions to transpositions. Hint: think about the sizes of the conjugacy classes of elements of order two.

3. Let G be a finite group. Cayley's theorem says that G is isomorphic to a subgroup of $\text{Perm}(G)$, where $\text{Perm}(G)$ is the group of all permutations of G. Let $\pi : G \to \text{Perm}(G)$ be the homomorphism from G into $\text{Perm}(G)$.

   (a) Let $|G| = n$ and let $x \in G$ with $|x| = m$. Describe the cycle structure of the permutation $\pi(x)$. (Here $|x|$ denotes the order of the element x).

   (b) Prove that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

   (c) Prove that if G contains an element x with $|x|$ even and $|G|/|x|$ odd, then G has a subgroup of index 2 and, thus, G is not simple.

4. Let $a = \sqrt{2}\omega \in \mathbb{C}$, where $\omega = e^{2\pi i/3}$.

   (a) Find the minimal polynomial for a over $\mathbb{Q}$.

   (b) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}$ (justify your answer).

5. Let $F \subseteq E$, where E is a field extension of F and $|F| = q < \infty$. Show that $F = \{\alpha \in E \mid \alpha^q = \alpha\}$. Hint: To show "$\subseteq$," use the fact that $F \setminus \{0\}$ is a finite group, and to show "$\supseteq$," count.

6. Let $F \subseteq E$ be a field extension with $|E : F|$ finite, and let $\alpha \in E$.

   (a) Recall that E is a vector space over F. Define $T_\alpha : E \to E$ by $T_\alpha(x) = \alpha x$ (multiplication by α). Show that T_α is a linear transformation (the scalars are from F so it is an F-linear transformation).

   (b) Show that α is a root of the characteristic polynomial of T_α. (*Recall that the characteristic polynomial of a matrix M is* $\det(\lambda I - M)$).

   (c) Given that $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is a basis for $\mathbb{Q}(1 + \sqrt[3]{2} + \sqrt[3]{4})$ over $\mathbb{Q}$, use this method to find a polynomial of degree 3 satisfied by $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

7. An element m of the R-module M is a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}$$

   (a) Prove that if R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M.

   (b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule.

   (c) Show that if R has zero divisors, then every nonzero R-module has torsion elements.

8. Suppose that K is a normal subgroup of a finite group G and that S is a Sylow p-subgroup of G. Prove that $K \cap S$ is a Sylow p-subgroup of K.

9. Let G be a group. Two subgroups H and J of G are *conjugate* in G if $gHg^{-1} = J$ for some $g \in G$. Let $F \subseteq E$ be a finite field extension. Two intermediate fields $F \subseteq K \subseteq E$ and $F \subseteq L \subseteq E$ are *F-isomorphic* if $\alpha(K) = L$ for some α in the Galois group $G(E/F)$.

   Let $F \subseteq E$ be a finite Galois extension. Let $F \subseteq K \subseteq E$ and $F \subseteq L \subseteq E$ be intermediate fields. Prove that K and L are F-isomorphic if and only if $G(E/K)$ and $G(E/L)$ are conjugate in $G(E/F)$.