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AUSLANDER SYSTEMS

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ABSTRACT. The authors generalize the dynamical system constructed by J. Auslander in 1959, resulting in perhaps the simplest family of examples of minimal but not strictly ergodic systems. A characterization of unique ergodicity and mean-L-stability is given. The new systems are also shown to have zero topological entropy and fail to be weakly rigid. Some results on the set of idempotents in the enveloping semigroup are also achieved.

1. INTRODUCTION

In 1958, J. Auslander [1] introduced a minimal, mean-L-stable, yet non distal dynamical system which is a subset of the unit square and projects onto the triadic adding machine on the unit interval. This example was inspired by an example due to E. E. Floyd [4]. Auslander's example was provided with the intent to illustrate possible behaviours of mean-L-stable systems, and due to its intriguing properties, earned interest in its own right.

The construction leading to Auslander's example naturally lends itself to generalization. In this paper, we study a class of generalizations of this example which yield minimal topological systems we term *Auslander systems*. These systems are easy to describe and not difficult to visualize, yet they exhibit nontrivial dynamics which are explored in this paper. Several results are achieved including of note: On entropy: in [2], Auslander and Berg show that the entropy of Auslander's original example is zero. We show that this remains true for all Auslander systems. On unique ergodicity and mean-L-stability: We prove that Auslander systems can either be uniquely ergodic, or fail to be so, and we give a characterization of when unique ergodicity holds. As a consequence, some Auslander systems are mean-L-stable and some are not. On the set of idempotents $J(X)$ in the enveloping semigroup: we compute $J(X)$ for a "well behaved" class of Auslander systems, resulting in a set of cardinality c (the largest cardinality $J(X)$ could achieve in general is 2^c), and give insight to the fact that in general Auslander systems, $J(X)$ could be quite complex (for a thorough and enlightning treatment on the role of the set of idempotents in the enveloping semigroup, see [3]). Despite the complexity of the enveloping semigroup for these systems, it is possible to prove that the identity function is never in the asymptotic enveloping semigroup of an Auslander system. So Auslander systems are never weakly rigid.

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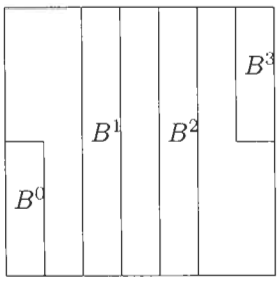
2. The construction

Let $B_0 = [0,1] \times [0,1]$. We say that α determines a subdivision pattern for B_0, given by a subset $\alpha(B_0)$ of B_0, if $\alpha(B_0)$ is a union of subrectangles of B_0, denoted by $B^0, B^1, \ldots, B^{n-1}$, such that for every superscript i,

- The width of B^i equals $(2n-1)^{-1}$,
- B^i is centered about the axis $x = \frac{4i+1}{2(2n-1)}$,
- B^i meets the axis $y = 0$ or $y = 1$,
- B^i has height 1 or $\frac{1}{2}$.

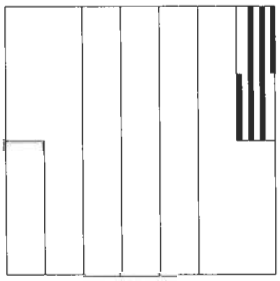
Say that B^i is of type F (for full height) if the height of B^i is 1 and of type H (for half height) if the height of B^i is $\frac{1}{2}$.

We can then define α on any rectangle $R \subset B_0$ by superimposing $\alpha(B_0)$ onto R. This yields a union of subrectangles of R which again can be described as of type F or H by analogy to $\alpha(B_0)$. For example, if $\alpha(B_0)$ is given by:



Unit Square

then $\alpha(B^3)$ is given by the shaded portion:



In the above figure, the two middle shaded rectangles are of type F.

A fixed sequence $\{\alpha_i\}_{i \in \mathbf{N}}$ of subdivision patterns determines a dynamical system (X,T) in the following inductive manner: let n_i be the number of rectangles in $\alpha_i(B_0)$ and set $\nu_j = \prod_{k=1}^{j} n_k$. Let $B_1 = \alpha_1(B_0) = B_1^0 \cup B_1^1 \cup \cdots \cup B_1^{n_1-1}$. Denote $\alpha_j(B_{j-1}^k)$ by $B_j^k \cup B_j^{k+\nu_{(j-1)}} \cup \cdots \cup B_j^{k+(n_j-1)\nu_{(j-1)}}$ and then set $B_j = \cup_{k=0}^{\nu_{(j-1)}-1} \alpha_j(B_{j-1}^k)$, which can also be denoted, using the above, by $\cup_{k=0}^{\nu_j - 1} B_j^k$. Notice that the superscripts of these rectangles B_j^k no longer increase successively from

left to right, although B_j^0 is always the leftmost rectangle and $B_j^{\nu_j-1}$ the rightmost. Finally, define T_j on B_j, $j \in N$, as the linear map which sends B_j^k to B_j^{k+1} for $k < \nu_j - 1$ and leaves T_j undefined on $B_j^{\nu_j-1}$.

Let $X = \cap_{i\in N} B_i$. X is a compact space which is made up of lines and points, and inherits the Euclidean metric of the plane. From this point on, the term "line" will always be taken to mean a line with nonzero length, and the term "degenerate point" will be taken to mean a point in X not lying on any line.

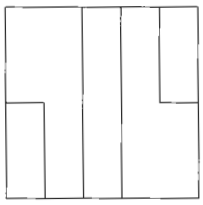
In order to define a continuous injective map T on X by $T = \lim_{i\to\infty} T_i$, we need to set some restrictions on our subdivision patterns. Continuity necessitates that T map the line or degenerate point with x-coordinate 1 to the line or degenerate point with x-coordinate 0. For T to be injective and well defined, X must then have a line over $x = 1$ if and only if X has a line over $x = 0$. If these two are indeed lines, then to ensure continuity there must exist a K such that for all $i \geq K$, B_i^m is of type F for all m. Otherwise, α_i has both its first and last boxes of full length for $i \geq K$ but has other subboxes of half length, in which case one can find two sequences of points (x, y) converging to $(1, y)$, one with (x, y) lying on a subbox half the length of the other. The images under T of these two sequences converge to different points. This violates the continuity of T. Note that the assumption that B_i^m is of type F for all $i \geq K$ and all m yields a system (X, T) isomorphic to the direct product of the adding machine and [0,1]. Also, if there are infinitely many i with B_i^m of type H for all m, then (X, T) will again be isomorphic to the adding machine. Since the dynamics of the adding machine are well known, we omit these two cases in our study.

Definition 2.1. If a sequence of subdivision patterns $\{\alpha_i\}_{i\in N}$ satisfies the properties:

- there are infinitely many i with B_i^0 of type H and infinitely many j with $B_j^{\nu_j-1}$ of type H,
- at most finitely many i have B_i^m of type H for all m,

then the resulting system (X, T) will be called an Auslander system.

Auslander systems are generalizations of the dynamical system introduced by J. Auslander in [1]. In the original Auslander example, all α_i are the same and determine the subdivision pattern shown below:



Unit Square

We may describe the points and lines of X symbolically. Let $N(i) = \{0, 1, ..., n_i - 1\}$ be a numbering of the subboxes of $\alpha_i(B_0)$ from left to right, and let $Q(i) \subset N(i)$ correspond to those subrectangles of type F. If $(x, y) \in X$, we may describe x uniquely by $x_1x_2x_3..... \in Y = \prod_{i=1}^{\infty} N(i)$ by setting $x_j = k \Leftrightarrow (x, y)$ belongs to the $(k + 1)$ subrectangle of $\alpha_j(B_{j-1}^m)$. We will denote $x_1x_2x_3\ldots$ resulting from x by $\overline{x}$. Also, if L is a line in X with x-coordinate x, we will have occasion to write $L \sim x_1x_2x_3\ldots$. The map T on X projects to the adding machine on Y, modulo n_i at slot i, carrying to the right.

The following two lemmas are consequences of the construction.

Lemma 2.2. *Let (X,T) be an Auslander system. Let $(x,y) \in X$ and let $\overline{x} = x_1x_2x_3\ldots$. Then (x,y) belongs to a line iff there exists k such that for every $i \geq k$, $x_i \in Q(i)$.*

Lemma 2.3. *Let (X,T) be an Auslander system. X contains only countably many lines if and only if there exists K such that for every $i \geq K$, $|Q(i)| = 1$.*

Now denote $Q(i)^c$ by $H(i)$ and divide $H(i)$ into $H_L(i)$ and $H_U(i)$, depending on whether the corresponding rectangle of $\alpha_i(B_0)$ lies in the lower or the upper half of B_0 respectively. Thus $N(i) = Q(i) \cup H_L(i) \cup H_U(i)$. The following two lemmas follow from writing x in the symbolic form $\overline{x}$.

Lemma 2.4. *If $\{i : x_i \in H(i)\}$ is finite, then (x,y) belongs to a line L whose length $l(L)$ is $\frac{1}{2^k}$ iff $|\{i : x_i \in H(i)\}| = k$. If $\{i : x_i \in H(i)\}$ is infinite, then (x,y) is a degenerate point.*

Lemma 2.5. *If (x,y) is a degenerate point, then y can be found using $\overline{x}$ as stated below, and if (x,y) lies on line L, then the ordinate y_b of the base (bottom point) of L can be found in the same way: delete from $\overline{x}$ all x_j such that $x_j \in Q(j)$ and reindex the remaining sequence by N.*
Let $M = \{i \in N : x_i \in H_U(i)\}$. Then y (or y_b) equals $\sum_{n\in M} \frac{1}{2^n}$.

3. Minimality

Recall that for all points $(x,y) \in X$ except those with $\overline{x} = (n_1 - 1)(n_2 - 1)\ldots$, $T(x,y)$ is determined by some T_m.

Theorem 3.1. *Let (X,T) be an Auslander system. Then T is continuous on X.*

Proof. If there exists i such that $x_i \neq n_i - 1$ (so (x,y) is not in the last rectangle at every stage), then at stage i, (x,y) belongs to some B_i^k which maps linearly under T_i to rectangle B_i^{k+1}. Clearly T is then continuous at (x,y).

Otherwise $x = 1$, and $(1,y)$ is a degenerate point. Let J be such that B_J^0 and $B_J^{\nu_J - 1}$ have height less than ϵ. Let $\delta = \min\{\epsilon, \text{width of } B_J^i\}$. Then the $\delta \times \delta$ box surrounding (x,y) is mapped within the $\epsilon \times \epsilon$ box surrounding $T(x,y)$. □

Since (X,T) projects to the adding machine, we can follow the orbit of the x-coordinate of a point by using the adding machine on Y. Proximality is thus easy to determine:

Theorem 3.2. *Two points (x_1,y_1) (x_2,y_2) in an Auslander system are proximal iff $x_1 = x_2$ (and thus they belong to the same line).*

The next lemma and theorem will show that all Auslander systems are minimal. We will denote the usual Euclidean distance on X by $d(\ ,\ )$, and if L is a line in X, we will use $y_b(L)$ to denote the y-coordinate of the bottom point of L.

Lemma 3.3. *Let (X,T) be an Auslander system satisfying the following condition:*
(C) *Given a point on a line and a neighborhood U of this point, we can find a degenerate point $(x,y) \in X$ such that $(x,y) \in U$.*
Then (X,T) is a minimal system.

Proof. Let (x, y) be an arbitrary point in X. We will show that the orbit closure of (x, y) under T is all of X.

The orbit of (x, y) visits the neighborhood of any degenerate point since adding machines are minimal.

Otherwise, suppose that (x', y') is a point on a line and let U be a neighborhood of (x', y'). By condition (C), there exists a degenerate point (x_1, y_1) in U. But as before, the orbit of (x, y) comes arbitrarily close to (x_1, y_1). □

Theorem 3.4. *Let (X, T) be an Auslander system. Then (X, T) is minimal iff both of the following are satisfied:*

i) *There exists infinitely many i with $|H_U(i)| > 0$.*

ii) *There exists infinitely many j with $|H_L(j)| > 0$.*

Proof. If the condition does not hold then either exactly one or neither of i) and ii) is satisfied. If neither is satisfied, then (X, T) is not an Auslander system and besides, is clearly not minimal.

If i) is satisfied but not ii), then there will be a height $y_0 > 0$, below which no degenerate points exist. The set of degenerate points would then form a nonempty proper T-invariant subset of X. A similar situation arises if ii) is satisfied but not i).

Now assume i) and ii) are satisfied. To show (X, T) is minimal we just need to show it satisfies condition (C) of Lemma 3.3.

Let (x, y) belong to line L. Let U be a $p \times p$ neighborhood of (x, y) and $\overline{x} = x_1x_2x_3\ldots$. Choose (x', y') so that $\overline{x'} = x'_1x'_2x'_3\ldots$ and $x'_i = x_i$ for $1 \le i \le I$, where I is such that $\mathrm{d}(x, x') < p$. Choose $x'_i \in H_L(i) \cup H_U(i)$ for $i > I$ in such a way that, by Lemma 2.5, (x', y') is a degenerate point in U. □

4. Unique ergodicity and topological entropy

Recall that the adding machine is uniquely ergodic and has entropy zero.

Thus any interesting dynamical phenomena must occur on the set of lines $\tilde{L}$ of (X, T). Any ergodic measure μ on (X, T) must project to the unique measure which gives each symbol equal weight. If $\mu(\tilde{L}) = 0$ then X is measurably isomorphic to the adding machine and trivially (X, T) is uniquely ergodic and $h_\mu(T) = 0$. Note that $\tilde{L}$ is an invariant set; thus $\mu(\tilde{L}) = 0$ or 1 for any ergodic measure. We first show that $\mu(\tilde{L}) = 1$ is possible.

Divide $\tilde{L}$ into $\cup_k \tilde{L}_k$ where $\tilde{L}_k = \{L : L$ is a line associated with x-coordinate $x_1x_2\ldots$ such that k is the smallest integer with $x_i \in Q(i)$ for every $i \ge k\}$. Then $\mu(\tilde{L}) = \sum_k \mu(\tilde{L}_k)$. But $\mu(\tilde{L}_k)$ coincides with the measure of the projection of $\tilde{L}_k$ which is $\frac{|H(k-1)|}{|N(k-1)|} \prod_{i=k}^{\infty} \frac{|Q(i)|}{|N(i)|}$. Thus $\mu(\tilde{L}) = \sum_{k=1}^{\infty} (\frac{|H(k-1)|}{|N(k-1)|} \prod_{i=k}^{\infty} \frac{|Q(i)|}{|N(i)|})$ for every possible measure. Then $\mu(\tilde{L}) = 1$ if and only if there exists a k with $\prod_{i=k}^{\infty} \frac{|Q(i)|}{|N(i)|} > 0$, which is equivalent to the existence of a k with $\sum_{i=k}^{\infty} (1 - \frac{|Q(i)|}{|N(i)|}) < \infty$. For example, a sequence of patterns $\{\alpha_i\}$ with $|N(i)| = 2^i$ and $|Q(i)| = 2^i - 1$ will yield a dynamical system having $\mu(\tilde{L}) = 1$.

Theorem 4.1. *An Auslander system is uniquely ergodic if and only if $\mu(\tilde{L}) = 0$.*

Proof. We need to show that (X, T) is not uniquely ergodic if $\mu(\tilde{L}) = 1$. This can be done by defining a family of mutually singular measures on (X, T). Let $0 \le a \le 1$ and define f_a to be the map from the adding machine to (X, T) which takes a point

x to the point (x, y_a), where y_a lies on the line above x, with proportion a from the bottom of this line. If there is no line above x, then define f_a in the obvious way. Then f_a gives a measurable isomorphism between the adding machine and (X, T, μ_a) where the support of μ_a can be thought of as a cross section of (X, T). Clearly $\text{supp}(\mu_{a_1}) \neq \text{supp}(\mu_{a_2})$ unless $a_1 = a_2$, and each μ_a is ergodic. □

Definition 4.2. A compact dynamical system (X, T) is said to be mean-L-stable if for every pair of positive numbers ϵ_1 and ϵ_2, there is a positive number δ such that $x, y \in X$ with $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \epsilon_1$ for all n except in a set of upper density less than ϵ_2.

Remark. It follows from Theorem 4.1 and from results of [1] that minimal Auslander systems are uniquely ergodic if and only if they are mean-L-stable. More specifically, Theorem 8 of [1] implies that a minimal mean-L-stable system must be uniquely ergodic; and conversely, the proof given on pages 578-579 of [1] (which shows that Auslander's original example is mean-L-stable) generalizes to all Auslander systems with $\mu(\tilde{L}) = 0$.

We will compute the topological entropy $h(T)$ of an Auslander system (X, T) as the maximum of the measure theoretic entropies $h_\mu(T)$.

Theorem 4.3. *Let (X, T) be an Auslander system. Then $h(T) = 0$.*

Proof. The measures μ_a constructed in the proof of Theorem 4.1 are the only ergodic measures on (X, T). Also, $h_{\mu_a}(T) = 0$ for all measures μ_a. By the variational principle, $h(T) = 0$. □

5. Idempotents in the enveloping semigroup

The enveloping semigroup of a ($\mathbf{Z}$-action) dynamical system (X, T) is defined as

$$E(X, T) = E(X) = \overline{\{T^n : n \in \mathbf{Z}\}}$$

where the closure is taken in the product topology. The asymptotic enveloping semigroup of (X, T) is defined to be $E(X) - \{\text{isolated points of } E(X)\}$ and is denoted by $H(X)$. The set of idempotents $J(X)$ in $H(X)$ is never empty (this is a consequence of what is sometimes called the Ellis theorem, see for instance [3]), and can sometimes be quite complex, as can be the enveloping semigroup itself. We can completely determine $J(X)$ for the subclass of Auslander systems having only countably many lines. The complexity of the general situation is then indicated by an example; however, the state of rigidity can be settled for all situations.

Thus we now consider minimal (X, T) with the property that $|Q(k)| = 1$ for every $k \geq K$, for some K. The analysis of $J(X)$ is simpler here because the behavior of $f \in H(X)$ on one line is mimicked by its behavior on every other line. We first define what it means for two pairs of lines to have the same relative position, then show why we can find maps $f \in H(X)$ which map all lines into certain subsets of themselves. Such maps can then be used to find the idempotents in $J(X)$.

Definition 5.1. Given four line segments L^1, L^2, L^3, L^4 in X, with lengths indicated by $l(L^i)$, we say that L^2 is to L^1 what L^4 is to L^3 if the following conditions hold:

a) $l(L^1) \cdot l(L^4) = l(L^2) \cdot l(L^3)$, and

b) if b_i denotes the height of the bottom point of L^i, for $i = 1, \ldots, 4$, then $(b_2 - b_1) l(L^3) = (b_4 - b_3) l(L^1)$.

In the following, let L_1 denote a line of maximal length.

Lemma 5.2. *Let (X,T) be an Auslander system with the property $|Q(k)| = 1$ for every k larger than some K. Let L be an arbitrary line in X. There exists a map $f \in H(X)$ such that f maps L_1 linearly onto a subline of L_1 which is of the same length and at the same height as L. Furthermore, such an f is unique as it must map any other line L_2 in X linearly to a subline of itself satisfying $f(L_2)$ is to L_2 what $f(L_1)$ is to L_1, and take every degenerate point to itself.*

Proof. Without loss of generality, we assume $l(L_1) = 1$, so $L_1 \sim q_1 q_2 \dots$ with $q_i \in Q(i)$ for every i. Write $L \sim n_1 n_2 \dots$. Let f be the limit of a subsequence of T^{m_i} where m_i is chosen so that

$$T^{m_i}(L_1) \sim q_1 q_2 \dots q_i n_1 n_2 \dots n_r q_{i+r+1} \dots .$$

Thus

$$m_i = (n_1 - q_{i+1})\nu_i + (n_2 - q_{i+2})\nu_{i+1} + \dots + (n_r - q_{i+r})\nu_{i+r-1}.$$

Then $f \in H(X)$ maps L_1 as specified in the statement of the theorem. If L_2 is an arbitrary line, $L_2 \sim s_1 s_2 \dots$, then since $|Q(k)| = 1$ for every k larger than K, there exists M such that $q_m = s_m$ for every $m \geq M$. So $T^{m_i}(L_2)$ eventually has the form $s_1 s_2 \dots s_i n_1 n_2 \dots n_r s_{r+i+1} \dots$, and $T^{m_i}(L_2)$ is to L_2 what L is to L_1. Finally, since T^{m_i} fixes the first i symbols of any expansion in Y, f will map every degenerate point in X to itself. □

Corollary 5.3. *Let (X,T) be an Auslander system with the property $|Q(k)| = 1$ for every k larger than some K. If $f \in H(X)$ and $f(L_1) = \{(x,y)\}$ with $(x,y) \in L_1$, then the image under f of any degenerate point in X is itself and the image under f of any line L is the singleton $\{(x',y')\}$ where $(x',y') \in L$ and where the position of y' on L is proportional to that of y on L_1.*

Proof. We can write f as $\lim_{i\to\infty} f_i$, $f_i \in E(X)$, where $f_i(L_1) = L_i \subset L_1$. Then given an arbitrary line L in X, by Lemma 5.2, $f_i(L)$ is to L what $f_i(L_1)$ is to L_1. Since $f_i(L) \to f(L)$, the result follows. □

Now let f_r be the element in $H(X)$ which takes L_1 to (x,r), the point on L_1 of height r. Then Corollary 5.3 determines f_r on the rest of X.

Theorem 5.4. *If $\{\alpha_i\}$ is a sequence of subdivision patterns producing a minimal Auslander system (X,T) and if there exists a K such that for every $k \geq K$, $|Q(k)| = 1$, then $J(X) = \{f_r : r \in [0,1]\}$.*

Proof. Since idempotents must project to idempotents, any idempotent in $H(X)$ must project to an idempotent on the adding machine. However, the adding machine is equicontinuous and so the only idempotent in its enveloping semigroup is the identity map. Thus if $f \in J(X)$ and L_1 is a line of maximum length, then either $f(L_1) = L_1$ or $f(L_1)$ is a singleton. The first case is impossible because $|Q(k)| = 1$ for every $k \geq K$ implies that there are at most finitely many lines of maximum length in X. Thus the identity map is not in the asymptotic enveloping semigroup.

If r is a dyadic rational then we can use Lemma 5.2 to find a map in $H(X)$ taking L_1 to (x,r) on L_1. Since $H(X)$ is closed and since the dyadic rationals in $[0,1]$ are dense, $f_r \in J(X)$ for all $r \in [0,1]$. □

Note that if the system is not minimal, then one of conditions (i) or (ii) of Theorem 3.4 is violated. If there exists only finitely many i with $|H_U(i)| > 0$, then the only lines arbitrarily close to L_1 are those with bottom point having zero y-coordinate. This makes it impossible for any idempotent other than f_0 to be in $H(X)$. Similarly, if only finitely many i have $|H_L(i)| > 0$, then $J(X) = \{f_1\}$.

Next we exhibit a subdivision pattern $\{\alpha_i\}$ resulting in an Auslander system (X, T) with the property that there exist $f \in J(X)$ and two lines L and L' in X both of length 1 such that $f(L)$ is a singleton on L and $f(L') = L'$. Thus it is possible that the set of idempotents for this system are much more varied than in the subclass discussed above. However, it is not the case that such a set is necessarily larger; in the example below it can be shown that f_0 (the map that takes every line to its bottom point) is not a member of $E(X)$. Thus not all Auslander systems satisfy $\{f_r : r \in [0,1]\} \subseteq J(X)$.

Let $\{\alpha_i\}$ be such that for every i, $N(i) = \{0,1,2,3\}$, $Q(i) = \{1,2\}$, $H_U(i) = \{3\}$ and $H_L(i) = \{0\}$. In the resulting system (X, T), consider the two lines L and L' given respectively by the expansions $11111\ldots$ and $212212212\ldots$. Let m_1 be such that $T^{m_1}(11111\ldots) = 11102011111\ldots$ and inductively, let m_i be such that $T^{m_i}(11111\ldots) = n_1 n_2 \ldots$ where $n_j = 1$ for the first $3i$ symbols, then the pattern 020 is repeated for the next $3i$ symbols and then $n_j = 1$ for all j after that. More specifically,

$$m_i = \textstyle\sum_{j=0}^{i-1}(-4^{3(i+j)+2} + 4^{3(i+j)+1} - 4^{3(i+j)}).$$

It is easy to check that $T^{m_i}(L)$ converges to the bottom point of L and that $T^{m_i}(L')$ converges to L' (for instance,

$$T^{m_1}(L') = 212121212\ldots, \qquad T^{m_2}(L') = 212212121121212212212\ldots$$

etc.). Furthermore, assuming that T^{m_i} converges on X (otherwise take a subnet), it is also easy to see that T^{m_i} projects to the identity on the adding machine since T^{m_i} leaves the first $3i$ symbols fixed in the expansion of any line or point in X. The limit of T^{m_i} is therefore an element of $J(X)$ taking L to a singleton point on L and L' to itself.

6. Rigidity

In [5], Glasner and Maon define three degrees of rigidity for topological systems, the two weakest being:

Definition 6.1. (X, T) is weakly rigid if $\forall \epsilon > 0$ and points $z_1, z_2, \ldots z_n \in X$, $\exists k \in \mathbf{Z} - \{0\}$ such that $d(T^k z_i, z_i) < \epsilon$ $(i = 1, \ldots, n)$.

Definition 6.2. (X, T) is rigid (w.r.t. some sequence $n_k \nearrow \infty$) if $T^{n_k} z \to z$ $\forall z \in X$.

Clearly Def. 6.1 is equivalent to

Definition 6.3. (X, T) is weakly rigid if $id_X \in H(X)$.

Glasner and Maon show that the topological entropy of a rigid flow is zero. Note that we have shown in section 4 that the topological entropy of any Auslander system is zero. We show in this section that Auslander systems fail to even be weakly rigid. Indeed, despite the complexity of $J(X)$ for a general Auslander system, we can still prove that id_X is never in $H(X)$.

Recall that T acts on the first coordinate as the adding machine. For $\overline{x} = x_1 x_2 \ldots$, let us write $\overline{Tx} = (x_1 + 1)x_2 \ldots$. Then $\overline{T^{\nu_1} x} = x_1(x_2 + 1)x_3 \ldots$, $\overline{T^{\nu_j} x} =$

$x_1 \dots x_j(x_{j+1}+1)x_{j+2} \dots$, etc. Given a positive integer n, we can write n uniquely as $c_0 + c_1\nu_1 + \cdots + c_j\nu_j$, and $\overline{T^n x} = (x_1 + c_o)(x_2 + c_1) \dots (x_{j+1} + c_j)x_{j+2} \dots$. All sums must be taken mod n_j and there might be 'carrying over' from one index to the next.

Theorem 6.4. *Auslander systems are not weakly rigid.*

Proof. Assume there are infinitely many k with $|Q(k)| > 1$ (otherwise section 5 gives the result). We will assume $id_X \in H(X)$ and reach a contradiction.

If $id_X \in H(X)$ then given two lines L_1, L_2 and arbitrary ϵ, there must be m such that $T^m(L_i)$ is within ϵ of L_i, $i = 1, 2$. Assume $m > 0$ (else switch the roles of $L_1$and L_2 in the following argument). Pick the two lines to be

$$L_1 \sim l_{11}l_{12} \dots, \qquad L_2 \sim l_{21}l_{22} \dots$$

such that l_{1i} is the smallest element in $Q(i)$ and l_{2i} the largest. (It is enough to take this for large enough i.) There could be i such that $l_{1i} = l_{2i}$, but there are infinitely many i where they are not equal, by assumption.

Write m as $c_j\nu_j + c_{j+1}\nu_{j+1} + \cdots + c_p\nu_p$ where $c_j \neq 0$. We know

$$T^{c_j\nu_j}(L_1) \sim l_{11} \dots l_{1j}(l_{1(j+1)} + c_j) \dots$$

and

$$T^{c_j\nu_j}(L_2) \sim l_{21} \dots l_{2j}(l_{2(j+1)} + c_j) \dots .$$

For m sufficiently large, $l_{1(j+1)} + c_j$ and $l_{2(j+1)} + c_j$ must be elements in $Q(j+1)$. Since $c_j > 0$, there must be carrying over in $T^{c_j\nu_j}(L_2)$, i.e.,

$$T^{c_j\nu_j}(L_2) \sim l_{21} \dots l_{2j}(l_{2(j+1)} + c_j)(l_{2(j+2)} + 1) \dots .$$

Now there are two possibilities:

a) If $B_{j+2}^{\nu_{j+2}-1}$ is of type H, then $l_{2(j+2)} + 1 \notin Q(j+2)$ and c_{j+1} must be nonzero. Thus m includes the summands $c_j\nu_j$ and $c_{j+1}\nu_{j+1}$. Note

$$T^{c_j\nu_j + c_{j+1}\nu_{j+1}}(L_2) \sim l_{21} \dots l_{2j}(l_{2(j+1)} + c_j)(l_{2(j+2)} + 1 + c_{j+1}) \dots$$

and we again have carrying over, now in the $j+2$ digit. Thus

$$T^{c_j\nu_j + c_{j+1}\nu_{j+1}}(L_2) \sim l_{21} \dots l_{2j}(l_{2(j+1)} + c_j)(l_{2(j+2)} + 1 + c_{j+1})(l_{2(j+3)} + 1) \dots .$$

b) If $B_{j+2}^{\nu_{j+2}-1}$ is of type F, then $l_{2(j+2)} = n_{j+2} - 1$ and whether c_{j+1} is zero or not, we have carrying over in the $j+2$ term.

Since there is carrying over in the next digit, we can repeat the above: either c_{j+2} is zero or not, but either way, there is carrying over to the next term. This process never stops and we know a) occurs infinitely many times. Thus m contains infinitely many $c_i \neq 0$, which contradicts it being finite. □

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