Projectional Entropy in Higher Dimensional Shifts of Finite Type

Aimee Johnson*
Department of Mathematics and Statistics,
Swarthmore College, Swarthmore, PA 19081

Steve Kass†
Kathleen Madden‡
Department of Mathematics and Computer Science,
Drew University, Madison, NJ 07940

Any higher dimensional shift space $(X, \mathbb{Z}^d)$ contains many lower dimensional shift spaces obtained by projection onto r-dimensional sublattices L of $\mathbb{Z}^d$ where $r < d$. We show here that any projectional entropy is bounded below by the $\mathbb{Z}^d$ entropy and, in the case of certain shifts of finite type satisfying a mixing condition, equality is achieved if and only if the shift of finite type is the infinite product of a lower dimensional projection.

1. Introduction

Higher dimensional shifts of finite type consist of arrays of symbols containing only certain allowed configurations. They are a key object of study in symbolic dynamical systems and find applications in information theory and in the study of global properties of cellular automata. One important property of a shift of finite type is its topological entropy; this provides a measure of the complexity of the system and is invariant under conjugacy. In an attempt to understand the subdynamics of a system, one can consider lower dimensional directional entropies such as those defined by Milnor [1]. Unfortunately, for higher dimensional shifts of finite type with positive entropy, the directional entropy is not helpful because it is always infinite. In this paper, we consider a more naïve directional entropy, namely the entropy of the lower dimensional shift space obtained by restricting the points in the $\mathbb{Z}^d$ shift space to a $\mathbb{Z}^r$ sublattice L where $r < d$. We call this the L projectional entropy of the $\mathbb{Z}^d$ shift space. Projectional entropy is related to Milnor's directional entropy in that the r-dimensional directional entropy of L is the supremum of the L projectional entropies within a conjugacy class. We

Electronic mail addresses: *aimee@swarthmore.edu, †skass@drew.edu, ‡kmadden@drew.edu.

Complex Systems, 17 (2007) 243–257;

will be concerned with investigating the infimum of the L projectional entropies within a conjugacy class.

The entropy of the $\mathbb{Z}^d$ shift space is a lower bound for its projectional entropies (Lemma 4.3) and thus the infimum of the projectional entropies within a conjugacy class is greater than or equal to the $\mathbb{Z}^d$ entropy. It is possible for the infimum of the projectional entropies to equal the $\mathbb{Z}^d$ entropy of the $\mathbb{Z}^d$ shift space. For example, consider the two-dimensional full shift on two symbols $\{0, 1\}$ where all horizontal and vertical transitions are allowed; the $\mathbb{Z}^2$ entropy and the one-dimensional projectional entropy in any direction are $\log 2$. For another example consider the two-dimensional shift on two symbols $\{0, 1\}$ where 11 is not allowed horizontally but every other transition is allowed. In this example, the $\mathbb{Z}^2$ entropy and the projectional entropy on the horizontal axis are both equal to $\log((1+\sqrt{5})/2)$ while the projectional entropies in all other directions are $\log 2$. We will call both of these examples, as the infinite cartesian product of the lower dimensional shift space obtained by projection onto $L = \{k\vec{e}_1 : k \in \mathbb{Z}\}$, degenerate (see Definition 2.2). In Theorem 4.1 we show that for an extendible, block strongly irreducible shift of finite type, the projectional entropy is equal to the $\mathbb{Z}^d$ entropy if and only if the system is degenerate.

In the next section, we review the basic terms needed for what follows. Further background details can be found in [2]. In section 3 we discuss entropies associated with higher dimensional symbolic systems. In section 4, we define projectional entropy and consider which projectional entropies are possible within a conjugacy class. We conclude in section 5 with a discussion of examples and open questions.

2. Background

Let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a finite alphabet and let $X^d_{[n]}$ be the compact metric space $\mathcal{A}^{\mathbb{Z}^d}$. For $x \in X^d_{[n]}$ and $\vec{v} \in \mathbb{Z}^d$, let $x_{\vec{v}}$ denote the symbol at position $\vec{v}$ in x. Let

$$\sigma_d : X^d_{[n]} \times \mathbb{Z}^d \to X^d_{[n]}$$

be the continuous $\mathbb{Z}^d$-action defined by

$$\sigma_d(x, \vec{v})_{\vec{w}} = x_{\vec{v}+\vec{w}}$$

for all $\vec{v}, \vec{w} \in \mathbb{Z}^d$ and all $x \in X^d_{[n]}$. We call σ_d the *d-dimensional shift map* and $(X^d_{[n]}, \sigma_d)$ the *d-dimensional full n-shift*. When it causes no confusion, we will denote the d-dimensional full n-shift by $(X_{[n]}, \sigma_d)$.

For any $\vec{v} = \langle v_1, v_2, \ldots, v_d \rangle \in \mathbb{Z}^d$ and $x \in X^d_{[n]}$, we say x is *$\vec{v}$-coordinate-wise periodic* if

$$\sigma_d(x, \langle q_1 v_1, q_2 v_2, \ldots, q_d v_d \rangle) = x$$

for any $q_i \in \mathbb{Z}$.

If X is a closed, shift invariant subspace of $X_{[n]}^{d}$, we call (X, σ_d) a d-dimensional *subshift* or *shift space*.

For $x \in X$ and $B \subset \mathbb{Z}^d$, we will denote the configuration of symbols appearing in x at the locations determined by B as x_B. We define $S(X, B) = \{x_B : x \in X\}$. In other words, $S(X, B)$ is all configurations occurring at the locations determined by B in any $x \in X$. A subset of $\mathbb{Z}^d$ of particular interest is the "rectangle" with side lengths $m_1, m_2, \ldots, m_d$:

$$B^{+}_{m_1, m_2, \ldots, m_d} = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : 0 \leq a_i < m_i \text{ for } 1 \leq i \leq d\}.$$

If $m_1 = m_2 = \cdots = m_d = m$, we will denote this "square" by B^{+}_{m}. Similarly, we will denote the "square" of side length $(2m - 1)$ that is centered at the origin as

$$B_m = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : |a_i| < m \text{ for } 1 \leq i \leq d\}.$$

A subshift (X, σ_d) is called a *d-dimensional shift of finite type* if it is defined by a list of allowable configurations on B_m for some $m > 0$. We will call a configuration of symbols on an arbitrary set $B \subset \mathbb{Z}^d$ *allowable* if all configurations on subsets $B_m + \vec{v} \subseteq B$ are allowable.

A *block map* $\phi : X \to Y$ between shift spaces (X, σ_d) and (Y, σ_d) is defined by a mapping Φ between $S(X, B_m)$ for some m and the symbols occurring in Y. Given $\Phi : S(X, B_m) \to \mathcal{A}$, where $\mathcal{A}$ is the alphabet for Y, the block map is then given by $\phi(x)_{\vec{v}} = \Phi(x_{\vec{v}+B_m})$ for all $\vec{v} \in \mathbb{Z}^d$. Maps between shift spaces are continuous and commute with the shift map if and only if they can be defined in this way. If a block map is onto, it is called a *factor map*, and we say that (X, σ_d) factors onto (Y, σ_d). If a factor map is one-to-one, it is called a *conjugacy* and we say that (X, σ_d) and (Y, σ_d) are conjugate. Conjugate shift spaces exhibit identical dynamical properties.

In two dimensions, conjugacies between shifts of finite type can always be decomposed into a finite sequence of vertical and horizontal out- and in-splittings and amalgamations [3]. A vertical out-splitting is constructed as follows: For each $a \in \mathcal{A}$, define the vertical follower set of a, $\mathcal{P}(a)$, to be the set of symbols that can appear vertically above a in some element of X. That is,

$$\mathcal{P}(a) = \{b \in \mathcal{A} \mid x_{(0,0)}x_{(0,1)} = ab \text{ for some } x \in X\}.$$

Then for each $a \in \mathcal{A}$, create a partition of $\mathcal{P}(a)$ consisting of $k_a > 0$ partition elements. For $B = \{\vec{0}, \vec{e}_2\} \subseteq \mathbb{Z}^2$ where $\{\vec{e}_1, \vec{e}_2\}$ is the standard basis for $\mathbb{Z}^2$ we define

$$\Phi : S(X, B) \to \{a^i \mid a \in \mathcal{A} \text{ and } 1 \leq i \leq k_a\}$$

via

$$\Phi\begin{pmatrix} b \\ a \end{pmatrix} = a^i$$

if and only if b is in the i^{th} partition element of $\mathcal{P}(a)$. This block map defines a conjugacy from (X, σ_2) onto its image. A vertical in-splitting is defined similarly by partitioning the set of vertical predecessors for each $a \in \mathcal{A}$.

Horizontal out- and in-splittings are defined analogously, and the inverse of a vertical (horizontal) out- or in-splitting is called a vertical (horizontal) out- or in-amalgamation.

An example of a system conjugate to a d-dimensional shift space (X, σ_d) is its $(2m-1)^d$ higher block presentation, denoted $(X[m], \sigma_d)$, where $m \in \mathbb{N}$. $X[m] \subseteq (S(X, B_m))^{\mathbb{Z}^d}$ and thus the "symbols" in $X[m]$ are the blocks in $S(X, B_m)$. For each $x \in X$ we obtain $x[m] \in X[m]$ via $x[m]_{\vec{v}} = x_{\vec{v}+B_m}$; that is, $x[m]_{\vec{v}}$ is the configuration appearing in x in the $(2m-1)^d$ block centered at $\vec{v}$.

There are other shift spaces related to (X, σ_d) that are of interest. One such space is the d-dimensional subshift constructed from (X, σ_d) using a finite cartesian product. Define (X^k, σ_d) where $k \in \mathbb{N}$ as

$$X^k = X \times X \times \cdots \times X$$

where $X \times X$ denotes the usual cartesian product and where

$$\sigma_d((x_1, x_2, x_3, \ldots, x_k), \vec{v}) = (\sigma_d(x_1, \vec{v}), \sigma_d(x_2, \vec{v}), \ldots, \sigma_d(x_k, \vec{v})).$$

For example, if $d = 1$ we can think of each element of X^k as a vertical stack of k bi-infinite sequences from X.

Another related shift space is a projection of (X, σ_d).

Definition 2.1. Let (X, σ_d) be a d-dimensional shift space and let $\mathcal{V} = \{\vec{v}_1, \ldots, \vec{v}_r\}$, $0 < r < d$, be linearly independent integral vectors in $\mathbb{Z}^d$. If $L \subset \mathbb{Z}^d$ is the subspace spanned by integer multiples of the vectors in $\mathcal{V}$, we let $X_L = S(X, L)$ be the set of arrays in X restricted to L. The **projection of** (X, σ_d) **onto** L, denoted (X_L, σ_r), is the $\mathbb{Z}^r$ subshift we obtain by identifying $\mathcal{V}$ with the standard generators $\{\vec{e}_1, \ldots, \vec{e}_r\}$ of $\mathbb{Z}^r$.

We will be particularly interested in sets $\mathcal{V}$ as defined above when there exists $\mathcal{U} = \{\vec{v}_{r+1}, \ldots, \vec{v}_d\}$ such that $\mathcal{V} \cup \mathcal{U}$ is a linearly independent set of integral vectors whose integer span is $\mathbb{Z}^d$. In this case we will call L an r-dimensional *sublattice* of $\mathbb{Z}^d$. Let $L' \subset \mathbb{Z}^d$ be the integer span of $\mathcal{U}$. Given X_L as defined above, we can create a d-dimensional subshift $X_L^{\mathbb{Z}^{d-r}}$ whose elements are the d-dimensional arrays of symbols achieved by associating to each location in L' a point from X_L; that is,

$$X_L^{\mathbb{Z}^{d-r}} = \left\{\bar{x} = \{x(\vec{u})_{\vec{u} \in L'} \mid x(\vec{u}) \in X_L\}\right\}.$$

To find the symbol at location $\vec{w} \in \mathbb{Z}^d$ in a point $\bar{x} \in X_L^{\mathbb{Z}^{d-r}}$, decompose $\vec{w}$ as $\vec{w} = \vec{u} + \vec{v}$, $\vec{u} \in L'$, $\vec{v} \in L$ and take the symbol in position $\vec{v}$ of the element $x(\vec{u})$:

$$\bar{x}_{\vec{w}} = x(\vec{u})_{\vec{v}}.$$

We define a d-dimensional action on $X_L^{\mathbb{Z}^{d-r}}$ by

$$\sigma_d(\vec{x}, \vec{w})_{\vec{w}'} = x(\vec{u} + \vec{u}')_{\vec{v}+\vec{v}'},$$

where $\vec{w} = \vec{v} + \vec{u}$, $\vec{w}' = \vec{v}' + \vec{u}'$ with $\vec{v}, \vec{v}' \in L$ and $\vec{u}, \vec{u}' \in L'$.

This shift space, although technically d-dimensional, is in some sense a trivial extension of a lower-dimensional space. Thus we are led to the following definition.

Definition 2.2. A d-dimensional shift space (X, σ_d) is **degenerate** if there exists a sublattice $L \subset \mathbb{Z}^d$ such that $X = X_L^{\mathbb{Z}^{d-r}}$.

While it may be difficult to determine if a shift space actually is degenerate, given a sublattice L it is often easy to determine that a shift space X is *not* equal to $X_L^{\mathbb{Z}^{d-r}}$. This can be done by counting coordinate-wise periodic points of various periods. To see this in the case where $d = 2$ and $r = 1$, let $m \in \mathbb{N}$. If $(X, \sigma_2) = (X_L^{\mathbb{Z}}, \sigma_2)$ where $L = \{k\vec{e}_1 : k \in \mathbb{Z}\}$, then it is clear that for any $n \in \mathbb{N}$,

$$|\{\langle m, n\rangle\text{-coordinate-wise periodic points of } X\}| = |\{m\text{-periodic points of } X_L\}|^n.$$

If a constant c does not exist for which the number of $\langle m, n\rangle$ coordinate-wise periodic points of (X, σ_2) is equal to c^n for all $n \in \mathbb{N}$, then X is not equal to $X_L^{\mathbb{Z}}$. Because the number of coordinate-wise periodic points is preserved by conjugacy, in this case we also know that $\tilde{X} \neq \tilde{X}_L^{\mathbb{Z}}$ for any $\tilde{X}$ conjugate to X.

In the literature (e.g., [4, 5]), a shift space is said to be *strongly irreducible* if there is an $s > 0$ such that for any two configurations x_B and $x'_{B'}$ occurring in X where B, $B' \subset \mathbb{Z}^d$ with the distance between B and B' greater than s, there exists $y \in X$ with $y_B = x_B$ and $y_{B'} = x'_{B'}$. It can be difficult to verify that a shift space is strongly irreducible, and we do not need all the power of strong irreducibility. Thus, we introduce a weaker mixing condition that is easier to verify, which we call *block strongly irreducible*.

Definition 2.3. A shift space is **block strongly irreducible** if there is an $s > 0$ such that for any two configurations $x_{B_m+\vec{v}}$ and $x'_{B_{m'}+\vec{v}'}$ occurring in X on blocks $B_m + \vec{v}$ and $B_{m'} + \vec{v}' \subset \mathbb{Z}^d$ with the distance between $B_m + \vec{v}$ and $B_{m'} + \vec{v}'$ greater than s, there exists $y \in X$ with $y_{B_m+\vec{v}} = x_{B_m+\vec{v}}$ and $y_{B_{m'}+\vec{v}'} = x'_{B_{m'}+\vec{v}'}$.

Block strong irreducibility is implied by square mixing as defined in [6] where it is shown (Example 3) that square mixing need not imply strong irreducibility. Thus not all block strongly irreducible shift spaces

are strongly irreducible. It is not difficult to verify that block strong irreducibility is preserved under conjugacy.

We note that when X is block strongly irreducible, there are only a finite number of sublattices L for which $X_L^{\mathbb{Z}^{d-r}}$ can have entropy other than $\log|\mathcal{A}|$. In this case, it is not difficult to determine if X is degenerate.

We close this section by noting that difficulties arise in higher dimensions that do not occur in the traditional one-dimensional case. For example, given a set of one-dimensional allowed blocks, it is relatively easy to determine whether the corresponding one-dimensional shift of finite type is nonempty. In higher dimensions, the question of whether there are *any* arrays of symbols given by a set of allowed blocks is referred to as the *nonemptiness problem* and is, in general, undecidable [7, 8]. Our results apply to nonempty shift spaces and our main theorem applies only to shifts of finite type for which every allowed configuration on a "rectangle" actually occurs.

Definition 2.4. A shift of finite type (X, σ_2) is **extendible** if given any $B = B^+_{m_1,m_2,\ldots,m_d}$ every allowed configuration on B is in $S(X, B)$.

Extensive background material on one-dimensional shifts of finite type can be found in [2] or [9]. D. Lind and B. Marcus also provide a good overview of higher dimensional shifts in Chapter 13 of [2].

3. Entropy

Entropy describes the complexity of a dynamical system. For shift spaces, intuitively it provides a measure of the growth rate of possible configurations in $S(X, B_m)$ as m increases.

Definition 3.1. The **entropy** of a d-dimensional symbolic dynamical system (X, σ_d), $d \geq 1$, is defined by

$$h(X) = \lim_{m\to\infty} \frac{1}{m^d} \log|S(X, B^+_m)|.$$

In fact, it is shown in [5] that for any sequence $\{\Xi_m\}_{m\in\mathbb{N}}$ of convex subsets of $\mathbb{Z}^d$ such that the inradii of the Ξ_m diverge to infinity,

$$h(X) = \lim_{m\to\infty} \frac{\log|S(X, \Xi_m)|}{|\Xi_m|}.$$

We note that entropy is a conjugacy invariant and thus for any d-dimensional shift space (X, σ_d), we have $h(X) = h(X[m])$. Lemmas 3.1 and 3.2 establish the relationship between the entropy of a finite or infinite cartesian product of a shift space and the entropy of the original shift space.

Lemma 3.1. For any d-dimensional shift space (X, σ_d), $h(X^k) = kh(X)$.

Proof. This follows easily from the definition of entropy and the fact that

$$|S(X^k, B_m^+)| = |S(X, B_m^+)|^k. \blacksquare$$

Lemma 3.2. If (X, σ_d) is degenerate with $X = X_L^{\mathbb{Z}^{d-r}}$ then $h(X) = h(X_L)$.

Proof. Let $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ and $\mathcal{U} = \{\vec{v}_{r+1}, \ldots, \vec{v}_d\}$ be the integral bases for L and L' as described previously. Consider the convex sets

$$\Xi_m = \left\{ \sum_{j=1}^{d} k_j \vec{v}_j \,|\, 0 \le k_j \le m \right\} \subseteq \mathbb{Z}^d$$

and

$$\bar{\Xi}_m = \left\{ \sum_{j=1}^{r} k_j \vec{v}_j \,|\, 0 \le k_j \le m \right\} \subseteq L.$$

Then

$$|S(X, \Xi_m)| = |S(X_L, \bar{\Xi}_m)|^{m^{d-r}},$$

and

$$\begin{aligned} h(X) &= \lim_{m\to\infty} \frac{\log |S(X, \Xi_m)|}{|\Xi_m|} \\ &= \lim_{m\to\infty} \frac{\log |S(X, \Xi_m)|}{m^d} \\ &= \lim_{m\to\infty} \frac{\log |S(X_L, \bar{\Xi}_m)|^{m^{d-r}}}{m^d} \\ &= \lim_{m\to\infty} \frac{\log |S(X_L, \bar{\Xi}_m)|}{m^r} \\ &= \lim_{m\to\infty} \frac{\log |S(X_L, \bar{\Xi}_m)|}{|\bar{\Xi}_m|} \\ &= h(X_L). \blacksquare \end{aligned}$$

We close this section by noting that when $d = 1$, entropies of many shift spaces, in particular shifts of finite type and factors of shifts of finite type, are easily calculated ([2], Chapter 4). However, when $d > 1$, although there are methods for obtaining entropy estimates for some shifts of finite type [10, 11], it is usually not feasible to compute entropy directly. Some notable exceptions can be found in [12] and [13].

4. Projectional entropy

Definition 4.1. Let (X, σ_d) be a d-dimensional shift space and let (X_L, σ_r) be a projection of (X, σ_d). Then the L **projectional entropy** of (X, σ_d) is $h(X_L)$.

For the remainder of this paper, we will use $d = 2$ and $L = \{k\vec{e}_1 : k \in \mathbb{Z}\}$ in order to simplify notation and arguments. We will denote X_L by X_1 and we will refer to $h(X_1)$ as the "horizontal projectional entropy." However, the following theorems can be generalized to any $d \in \mathbb{N}$ and any sublattice L.

Note that projectional entropy is not a conjugacy invariant. For example, the horizontal projectional entropy of the full shift on two symbols is $\log 2$, but the horizontal projectional entropy of the 3×3 higher block presentation of the full two shift is $\log 8$. In fact projectional entropy can rise, fall, or remain constant under a conjugacy as Lemma 4.1 demonstrates. In the statement of this lemma, X_2 denotes $X_{\bar{L}}$ where $\bar{L} = \{k\vec{e}_2 : k \in \mathbb{Z}\}$.

Lemma 4.1. Let (X, σ_2) be a shift of finite type and let

$$\phi : (X, \sigma_2) \to (Y, \sigma_2)$$

be a vertical out- or in-splitting. Then $h(Y_1) \geq h(X_1)$ and $h(Y_2) = h(X_2)$.

Proof. Assume that ϕ is a vertical out-splitting. Thus for each $x \in X$,

$$\phi(x)_{\vec{v}} = x^i_{\vec{v}}$$

where $x_{\vec{v}+\vec{e}_2}$ is in the i^{th} partition element of $\mathcal{P}(x_{\vec{v}})$.

Note that ϕ induces a conjugacy between the one-dimensional spaces (X_2, σ_1) and (Y_2, σ_1) and thus the vertical projectional entropy is unchanged [2].

Next consider horizontal configurations

$$x_0 x_{\vec{e}_1} x_{2\vec{e}_1} \cdots x_{(m-1)\vec{e}_1}$$

in $S(X_1, B^+_m)$. For each such configuration there corresponds at least one configuration in $S(Y_1, B^+_m)$. (There may be more than one corresponding configuration in $S(Y_1, B^+_m)$ depending on the number of allowable ways of vertically extending configuration $x_0 x_{\vec{e}_1} x_{2\vec{e}_1} \cdots x_{(m-1)\vec{e}_1}$.)

It follows that

$$|S(X_1, B^+_m)| \leq |S(Y_1, B^+_m)|$$

and $h(X_1) \leq h(Y_1)$ as desired.

The proof for an in-splitting is similar. ■

Note that Lemma 4.1 holds for horizontal out- or in-splittings with the roles of the vertical and horizontal directions reversed. Amalgamations, as the inverse of splittings, can lower projectional entropy or leave it constant. Lemma 4.2, the proof of which is left to the reader, states that for any (X, σ_d) for which $h(X) \neq 0$, by taking higher and higher block presentations, we can find conjugate systems with arbitrarily large horizontal projectional entropy.

Lemma 4.2. If (X, σ_2) has nonzero entropy, then

$$\lim_{n\to\infty} h(X[n]_1) = \infty.$$

For readers familiar with the notion of directional entropy [1], we briefly explain the relationship between the horizontal directional entropy and the horizontal projectional entropy for two-dimensional shifts of finite type. In this setting the definition of horizontal directional entropy reduces to

$$\sup_{n>0}\left(\limsup_{m\to\infty} \frac{1}{m} \log |S(X, B_{m,n})|\right).$$

Note that

$$h(X_1) = \limsup_{m\to\infty} \frac{1}{m} \log |S(X, B_{m,1})|$$

and thus the horizontal projectional entropy provides a lower bound for horizontal directional entropy. When $n > 1$ we have

$$h(X[n]_1) = \limsup_{m\to\infty} \frac{1}{m} \log |S(X, B_{m,n})|$$

and thus when $h(X) > 0$, Lemma 4.2 shows that horizontal directional entropy is infinite. We also note that in this setting, since

$$\begin{aligned}\infty &= \sup_{n>0}\left(\limsup_{m\to\infty} \frac{1}{m} \log |S(X, B_{m,n})|\right) \\ &= \sup_{n>0} h(X[n]_1) \\ &\le \sup\left\{h(Y_1) | (Y, \sigma_2) \text{ is conjugate to } (X, \sigma_2)\right\},\end{aligned}$$

horizontal directional entropy and the supremum of the horizontal projectional entropies in the conjugacy class of (X, σ_2) are both infinite and thus are equal.

For a fixed sublattice L, although the supremum of the L projectional entropies over members of a conjugacy class of a shift space with positive entropy is infinite, the L projectional entropy of each system in the conjugacy class is finite. In this work we are not interested in the supremum but in the infimum.

We first note that $h(X)$ serves as a lower bound for all projectional entropies and thus the infimum of the projectional entropies must be at least $h(X)$.

Lemma 4.3. Let (X, σ_2) be a two-dimensional shift space. Then $h(X) \le h(X_1)$.

Proof. Note that $S(X, B_m^+) \subseteq S(X_1, B_m^+)^m$, and thus

$$\begin{aligned} h(X) &= \lim_{m\to\infty} \frac{1}{m^2} \log|S(X, B_m^+)| \\ &\leq \lim_{m\to\infty} \frac{1}{m^2} \log(|S(X_1, B_m^+)|^m) \\ &= \lim_{m\to\infty} \frac{1}{m} \log|S(X_1, B_m^+)| \\ &= h(X_1) \end{aligned}$$

as desired. ■

So we are led to ask the following question: Given (X, σ_2), under what circumstances will $h(X)$ equal the horizontal projectional entropy? Theorem 4.1 answers this question for a significant class of subshifts. Example 5.3 shows that the theorem is not true without the strong irreducibility assumption.

Theorem 4.1. Let (X, σ_2) be an extendible, block strongly irreducible shift of finite type. Then $h(X) = h(X_1)$ if and only if (X, σ_2) is equal to $(X_1^{\mathbb{Z}}, \sigma_2)$.

Proof. If (X, σ_2) is equal to $(X_1^{\mathbb{Z}}, \sigma_2)$, then the fact that $h(X) = h(X_1)$ follows from Lemma 3.2. So suppose that

$$h(X) = h(X_1) = \log(\lambda).$$

Since (X, σ_2) is a shift of finite type, we may assume that it is defined via $n \times n$ allowed blocks for some $n \in \mathbb{N}$. Note that $X \subseteq (X_1)^{\mathbb{Z}}$. Suppose the claim is not true and $X \subsetneq (X_1)^{\mathbb{Z}}$. Then there exists a $k \times k$ block B which occurs in $(X_1)^{\mathbb{Z}}$ but not in X. (Note that block B is of the form $B_1 \times B_2 \times \cdots \times B_k$ where each B_i is a k block in X_1 but it is convenient to think of it as both a k block in X_1^k and a $k \times k$ block in $(X_1)^{\mathbb{Z}}$.)

Consider (X_1^k, σ_1). We show that (X_1^k, σ_1) is an irreducible sofic shift (where irreducible is as defined in [2] for one-dimensional shift spaces). First note that because (X, σ_2) is an extendible, block strongly irreducible shift of finite type, the higher block presentation $(X[n], \sigma_2)$ of (X, σ_2) is an extendible, block strongly irreducible shift of finite type as well. Thus both $(X[n]_1, \sigma_1)$ and $((X[n]_1)^k, \sigma_1)$ are easily shown to be one-dimensional irreducible shifts of finite type. There is a factor map

$$\phi : (X[n]_1, \sigma_1) \to (X_1, \sigma_1).$$

This factor map ϕ can be extended to a factor map from $((X[n]_1)^k, \sigma_1)$ to (X_1^k, σ_1). As the factor of an irreducible shift of finite type, (X_1^k, σ_1) is an irreducible sofic shift.

Let Y^k consist of sequences in X_1^k which do not contain B. So (Y^k, σ_1) is a proper subshift of (X_1^k, σ_1) and thus by Corollary 4.4.9 of [2] (p. 124) we have $h(X_1^k) > h(Y^k)$. From Lemma 3.1

$$h(X_1^k) = kh(X_1) = k\log(\lambda) = \log(\lambda^k).$$

Thus for some $\alpha < \lambda$

$$\log(\lambda^k) = h(X_1^k) > h(Y^k) = \log(\alpha^k).$$

Note that

$$|S(X, B_{km}^+)| \le |S((Y^k)^m, B_{km}^+)| = |S(Y^k, B_{km}^+)|^m.$$

It then follows that

$$\begin{aligned} h(X) = \log(\lambda) &= \lim_{m\to\infty} \frac{1}{(km)^2} \log|S(X, B_{km}^+)| \\ &\le \lim_{m\to\infty} \frac{1}{(km)^2} \log|S((Y^k)^m, B_{km}^+)| \\ &= \lim_{m\to\infty} \frac{1}{(km)^2} \log(|S((Y^k), B_{km}^+)|^m) \\ &= \frac{1}{k} \lim_{m\to\infty} \frac{1}{km} \log|S((Y^k), B_{km}^+)| \\ &= \frac{1}{k} h(Y^k) \\ &= \frac{1}{k} \log(\alpha^k) \\ &= \log(\alpha) < \log(\lambda). \end{aligned}$$

This is a contradiction and thus $X = X_1^{\mathbb{Z}}$ as desired. ■

5. Examples

We conclude with three examples and some open questions.

Example 5.1. Consider $(X_{[2]}, \sigma_2)$, the full shift on two symbols.

This example

1. is extendible and block strongly irreducible,
2. has two-dimensional entropy $\log 2$,
3. has projectional entropies of $\log 2$ on all sublattices L,
4. is degenerate with $(X_{[2]}, \sigma_2) = (X_{[2]_1^{\mathbb{Z}}}, \sigma_2)$.

Example 5.2. Let $Y(3) \subset X_{[3]}$ be the shift of finite type where $x \in Y(3)$ if and only if for any $\vec{v} \in \mathbb{Z}^2$, $x_{\vec{v}} + x_{\vec{v}-\vec{e}_1} + x_{\vec{v}-\vec{e}_2} \equiv 0, 1 \pmod 3$. That is, the sum of the symbols in any configuration of the following form must be zero or one (mod 3):

$$\begin{array}{cc} a & b \\ & c \end{array}$$

This example

1. is extendible and block strongly irreducible,
2. has two-dimensional entropy $\log 2$,
3. has projectional entropies of $\log 3$ on all sublattices L,
4. is not degenerate.

It is clear that Example 5.2 is extendible. It is also block strongly irreducible with $s = 1$. To see this, suppose that $\vec{v} \in \mathbb{Z}^2$ lies in two subsets S_1 and $S_2 \subset \mathbb{Z}^2$ of the form $S_i = \{\vec{w}_i, \vec{w}_i - \vec{e}_1, \vec{w}_i - \vec{e}_2\}$ for some $\vec{w}_i \in \mathbb{Z}^2$, $i = 1, 2$. Suppose further that we have an allowed configuration on $S_1 \cup S_2 \backslash \vec{v}$. Then there are two choices for the symbol in position $\vec{v}$ that result in an allowed configuration on S_1 and two choices for the symbol in position $\vec{v}$ that result in an allowed configuration on S_2. Because there are only three symbols occurring in $Y(3)$, there must be at least one choice that works for both S_1 and S_2. Using this fact, it is not difficult to see that any gap of width one between allowed configurations on two blocks can be filled in an allowable way. Thus this example is block strongly irreducible.

To see that $h(Y(3)) = \log 2$ we note that given any configuration of symbols in positions $(0, 0)$, $\vec{e}_1$ and $\vec{e}_2$, there are two allowed choices for the symbol in position $\vec{e}_1 + \vec{e}_2$. We say that $Y(3)$ has corner condition two and from this it clearly follows that

$$|S(X, B_m^+)| = 3^m 3^{m-1} 2^{(m-1)^2}$$

and $h(Y(3)) = \log 2$ as desired.

Any sequence of symbols from $\{0, 1, 2\}$ on a one-dimensional sublattice $L \subseteq \mathbb{Z}^2$ can be extended to a point in $Y(3)$ and thus $h(Y(3)_L) = \log 3$.

We can see that $(Y(3), \sigma_2)$ is not conjugate to $(X_{[2]}, \sigma_2)$ because $(Y(3), \sigma_2)$ has three fixed points while $(X_{[2]}, \sigma_2)$ has only two fixed points.

$(Y(3), \sigma_2)$ is not degenerate. For any sublattice L,

$$h(Y(3)_L^{\mathbb{Z}}) = \log 3$$

by property 3 above and Lemma 3.2, but $h(Y(3)) = \log 2$ by property 2 and thus by Theorem 4.1, $Y(3)$ is not equal to $Y(3)_L^{\mathbb{Z}}$.

The preceding argument does not allow us to conclude that a conjugate system will not be degenerate. However, given a specific sublattice L, counts of coordinate-wise periodic points might eliminate the possibility that a conjugate system $(\tilde{Y}, \sigma_2)$ will be of the form $(\tilde{Y}_L^{\mathbb{Z}}, \sigma_2)$. For example, let $L = \{k\vec{e}_1 : k \in \mathbb{Z}\}$. The reader can verify that $(Y(3), \sigma_2)$ has three $\langle 2, 1\rangle$ coordinate-wise periodic points (i.e., the fixed points), but there are fifteen $\langle 2, 2\rangle$ coordinate-wise periodic points, contrary to the observation spelled out after Definition 2.2. Thus $Y(3)$ is not equal to $Y(3)_1^{\mathbb{Z}}$ and, because conjugacy preserves coordinate-wise periodic point counts, any $\tilde{Y}$ conjugate to $Y(3)$ is not equal to $\tilde{Y}_1^{\mathbb{Z}}$.

Example 5.3 shows that Theorem 4.1 is not true without the block strong irreducibility assumption.

Example 5.3. Consider the two-dimensional shift of finite type $X \subseteq X_{[6]}$ given by horizontal and vertical transition rules as described by this adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This example

1. is extendible, but not block strongly irreducible,
2. has two-dimensional entropy $\log 2$,
3. has projectional entropy $h(X_1) = \log 2$,
4. is not equal to $X_1^{\mathbb{Z}}$.

It is not difficult to verify that $h(X_1) = \log 2$ and, because X has corner condition two, $h(X) = \log 2$. However X is not equal to $X_1^{\mathbb{Z}}$ since clearly there are arrays in $X_1^{\mathbb{Z}}$ which do not occur in X.

Example 5.3 is clearly not block strongly irreducible.

Within a conjugacy class, the L projectional entropies for a fixed sublattice L may vary. However, their infimum over the conjugacy class is trivially conjugacy invariant. If some member of the conjugacy class is degenerate for sublattice L, then this infimum is equal to the $\mathbb{Z}^2$ entropy. We are left with the following questions.

Open Questions. Let (Y, σ_2) be a block strongly irreducible shift of finite type for which no member of the conjugacy class is degenerate. Let I

denote the infimum of the L projectional entropies over *all* sublattices L and *all* systems conjugate to (Y, σ_2).

1. Is I achieved as a projectional entropy?
2. Is I greater than $h(Y)$?

We conjecture that in Example 5.2, I is equal to $\log 3$. If this is true, it would show that I can be achieved and can be bounded away from the full entropy. If that is the case, what can be said about a representative of the conjugacy class with the minimal projectional entropy? Does this representative somehow give us the clearest picture of the way in which the individual directions are interacting in the two-dimensional system?

References

[1] J. Milnor, "On the Entropy Geometry of Cellular Automata," *Complex Systems*, 2 (1988) 357–385.

[2] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding* (Cambridge University Press, Cambridge, 1995).

[3] A. Johnson and K. Madden, "The Decomposition Theorem for Two-Dimensional Subshifts of Finite Type," *Proceedings of the American Mathematical Society*, 127 (1999) 1533–1543.

[4] R. Burton and J. Steif, "New Results on Measures of Maximal Entropy," *Isreal Journal of Mathematics*, 89 (1995) 275–300.

[5] P. Ballister, B. Bollobás, and A. Quas, "Entropy Along Convex Shapes, Random Tilings and Shifts of Finite Type," *Illinois Journal of Mathematics*, 46 (2002) 781–795.

[6] S. Lightwood, "Morphisms from Non-periodic $\mathbb{Z}^2$ Subshifts II: Constructing Homomorphisms to Square Filling Mixing Shifts of Finite Type," *Ergodic Theory and Dynamical Systems*, 24 (2004) 1227–1260.

[7] R. Berger, "The Undecidability of the Domino Problem," *Memoirs of the American Mathematical Society*, 66 (1966).

[8] R. Robinson, "Undecidability and Nonperiodicity of Tilings of the Plane," *Inventiones Mathematicae*, 12 (1971) 177–209.

[9] B. Kitchens, *Symbolic Dynamics* (Springer-Verlag, New York, 1998).

[10] N. Markley and M. Paul, "Maximal Measures and Entropy for $\mathbb{Z}^r$ Subshifts of Finite Type," in *Classical Mechanics and Dynamical Systems* (Dekkar Notes, 70), edited by R. Devaney and Z. Nitecki (Dekkar, 1981).

[11] S. Friedland, "On the Entropy of $\mathbb{Z}^d$ Subshifts of Finite Type," *Linear Algebra and Its Applications*, 252 (1997) 199–220.

[12] D. Lind, K. Schmidt, and T. Ward, "Mahler Measure and Entropy for Commuting Automophisms of Compact Groups," *Inventiones Mathematicae*, 101(3) (1990) 593–629.

[13] P. Attard, "Markov Superposition for the Entropy and Correlation Functions in Two or Three Dimensions," *Series on Advances in Statistical Mechanics*, 14 (1999) 189–206.