Divergent Series

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Introduction

Traditionally, a series is said to "have a sum" if it is a convergent series, i.e., the partial sums converge. That is, $\sum a_n = A$ if

$$\lim_{n\to\infty} \sum_{i=0}^{n} a_i = A.$$

Of course, many series are not convergent. For example, the partial sums of $1-1+1-1\ldots$ are 1 at even indices and 0 at odd indices. But going back at least to Euler, it has been known that really only one number (if any) could serve as the sum of this series. For suppose $1-1+1-1\ldots = S$. Then $S = 1-1+1-1\ldots = 1-(1-1+1\ldots) = 1-S$, so $S = \frac{1}{2}$. So in a certain sense, $\frac{1}{2}$ "deserves" to be called the sum of this series. Mathematicians have thus extended the original notion of summation. Unfortunately, there is not one single extended summation method which is clearly better than the rest, so it is necessary to study several.

Definitions

Frequently summation methods assign the same sum for many of the series they sum. If a summation method P can sum all of the series summed by Q, and agrees with Q where they are both defined, then P is said to be *stronger* than Q. If P is stronger than Q and Q is stronger than P, then the summation methods P and Q are said to be equivalent. There are several properties which it is desirable for a summation method to possess. The first is that whenever a series converges, the method should give the same sum as ordinary convergence:

Definition A summation method P is *regular* if whenever $\sum a_n = A$ (by convergence), $\sum a_n = A\ (P)$.

This says that a summation method P is regular if it is stronger than the convergence method. Two other desirable properties also follow from the idea that summation methods extend the idea of convergence.

Definition A summation method P is *linear* if whenever $\sum a_n$ and $\sum b_n$ are summable by P, then so is $\sum pa_n + qb_n$, and $p\sum a_n + q\sum b_n = \sum pa_n + qb_n\ (P)$.

Definition A summation method P is *stable* if given any sequences a_n and $b_n = a_{n+1}$, $\sum a_n$ is summable by P if and only if $\sum b_n$ is summable by P, and $\sum b_n = (\sum a_n) - a_0$.

Convergent series satisfy all of these. The third, stability, is equivalent to the fact that finitely many terms of a convergent series may be rearranged, and the limit will be preserved.

Hölder Means

The "simplest method of summation of a divergent series" (Hardy 94) is to take averages of partial sums. This method makes use of the following proposition:

Proposition 0.1. *If a sequence $\{s_n\}_{n=0}^{\infty}$ converges, then the sequence*

$$\{\frac{s_0 + s_1 + \ldots + s_n}{n+1}\}_{n=0}^{\infty}$$

also converges, and converges to the same limit.

This is useful for summation because if a sequence of partial sums $\{s_n\}_{n=0}^{\infty}$ does not converge, the sequence of averages of the partial sums might. Thus we can define the $(H,1)$ sum of a series $\sum_{n=0}^{\infty} a_n$ to be the limit of the sequence of averages of partial sums. The above proposition ensures the regularity of this summation method, and it is also linear and stable.

Hölder Means (cont.)

By repeating this process of averaging initial segments of the sequence, this method can be generalized.

Definition Let $\sum_{n=0}^{\infty} a_n$ be a series. Define $H_n^0 = a_0 + a_1 + \ldots + a_n$. Then (inductively) define

$$H_n^{r+1} = \frac{H_0^r + H_1^r + \ldots + H_n^r}{n+1}.$$

If $\lim_{n\to\infty} H_n^k = L$ exists, then $\sum_{n=0}^{\infty} a_n$ is summable (H,k) to L. Note that summation (H,0) is then just convergence.

Theorem 0.2. *When $i \geq j$, the summation method (H,i) is stronger than the summation method (H,j). In other words, if $i \geq j$ and $\sum a_n = A\ (H,j)$, then $\sum a_n = A\ (H,i)$. In particular, each (H,j) is regular (this is the case of $i = 0$).*

Example

The series mentioned above, $1-1+1-1+\ldots$ is summable $(H,1)$. Then we have

$$\{H_n^0\} = (1,0,1,0,1,0,\ldots),$$
$$\{H_n^1\} = (\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \ldots)$$

with limit $\frac{1}{2}$.

Euler (left) and Hardy (right), both influential in the study of divergent series


Cesaro Sums

The (H,k) summations were defined by a process requiring $k+1$ summations and k divisions, which (while somewhat intuitive conceptually) can be difficult to work with. Cesaro sums will be preferable in that only one division is required.
Given a sequence $\sum a_n$, define

$$A_n^0 = A_n = a_0 + a_1 + \ldots + a_n.$$

Inductively, define

$$A_n^{k+1} = A_0^k + A_1^k + \ldots + A_n^k.$$

The terms A_n^k are then partial sums with many terms repeated; that is, they are larger than the initial partial sums. It is intuitive then that we should need to divide by something before taking the limit as n goes to infinity. Since we know the sequence $(1+0+0+\ldots)$ should sum to 1, there is (in some sense) only a single choice of what to divide by. Let E_n^k be the value that A_n^k takes for the sequence $(1+0+\ldots)$ (i.e., $a_0 = 1$ and other terms are 0). Then let

$$C_n^k = \frac{A_n^k}{E_n^k}.$$

Definition With C_n^k defined as above, we say that $\sum a_n = A\ (C,k)$ if $\lim_{n\to\infty} C_n^k = A$.

Theorem 0.3. *For each n, k,*

$$A_n^k = \sum_{v=0}^{n} \binom{n-v+k}{k} a_v = \sum_{v=0}^{n} \binom{v+k}{k} a_{n-v}.$$

The most important fact about Cesaro Sums is that they are equivalent to Hölder sums, despite the fact that when $k > 1$, the sequences $\{C_n^k\}_{n=1}^{\infty}$ and $\{H_n^k\}_{n=1}^{\infty}$ may be different:

Theorem 0.4. *Given a series $\sum a_n$, $\sum a_n = S\ (H,i)$ if and only if $\sum a_n = S\ (C,i)$.*

Cesaro Sums (cont.)

Because we have an explicit formula for the coefficients in the Cesaro Sums, most proofs are easier for them than for Hölder Means. For example, we can show that the Cesaro Sums are stable. Due to the equivalence with Hölder Means, the latter are stable as well.

Abel Sums

Another summation method is inspired by a Abel's Theorem that if a series $\sum a_n$ converges to A, then

$$\lim_{x\to 1^-} \sum a_n x^n = A.$$

This asserts that Abel sums are regular, if we define Abel Sums as:

Definition Given a series $\sum a_n$, if $\lim_{x\to 1} \sum a_n x^n$ exists, then this limit is defined to be the Abel (A) sum of the series.

In fact, we can generalize Abel's Theorem:

Theorem 0.5. *If $\sum a_n = A\ (C,k)$ for any k, then $\sum a_n = A\ (A)$.*

In the case of $k = 0$, this is Abel's Theorem.

Example

Using the same series $\sum(-1)^n$ as above, we get the power series

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x},$$

defined for $|x| < 1$. The limit as x approaches 1 from the left is defined and equals $\frac{1}{1+1} = \frac{1}{2}$, the same sum as found above.

Product Series

We can use Abel Sums to reach a classic result about product series. For finite series, products are unambiguously defined by the distributive law of arithmetic. But in the infinite case, there is more than one reasonable definition. The most common (inspired by power series), states that given series $\sum_0^{\infty} a_n$ and $\sum_0^{\infty} b_n$, the product series $\sum_0^{\infty} c_p$ is defined by

$$c_p = \sum_{m+n=p} a_m b_n.$$

If two series are conditionally convergent, their product series may not converge at all. But there are some partial results in this direction. We offer (without proof):

Theorem 0.6. *If series $\sum a_n = A$ and $\sum b_n = B$ are absolutely convergent, then so is the product series $(\sum a_n)(\sum b_n)$, and it converges to AB.*

Now we can prove:

Theorem 0.7. *If the series $\sum a_n$, $\sum b_n$, and $(\sum a_n)(\sum b_n) = \sum c_n$ (the product series) all converge to A, B, and C respectively, then $C = AB$.*

Proof. Since these series all converge, their terms must approach 0. This means that the power series $\sum a_n x^n$, $\sum b_n x^n$, and $\sum c_n x^n$ all converge absolutely for $|x| < 1$ and so, for $x \in [0,1)$,

$$(\sum a_n x^n)(\sum b_n x^n) = \sum c_n x^n$$

from Theorem 0.6. It follows from Abel's Theorem that the limit as x approaches 1 from the left exists, and so $AB = C$. □

References

1. Hardy, G. H. *Divergent Series.* Oxford: Clarendon Press, 1949.
2. Varadarajan, V. S. *Euler through time: a new look at old themes.* AMS, 2006.