4.1 Introduction

One of the central concepts of political science is power. While power itself is certainly many-faceted (with aspects such as influence and intimidation coming to mind), our concern is with the narrower domain involving power as it is reflected in formal voting situations (most often) related to specific yes–no issues. If everyone has one vote and majority rule is being used, then clearly everyone has the same amount of “power.” Intuition might suggest that if I have three times as many votes as you (and majority rule is still being used in the sense of “majority of votes” being needed for passage), then I have three times as much power as you have. The following hypothetical example should suffice to call this intuition (or this use of the word power) into question.

Suppose the United States approaches its neighbors Mexico and Canada with the idea of forming a three-member group analogous to the European Economic Community as set up by the Treaty of Rome in 1958. Recall that France, Germany, and Italy were given four votes each, Belgium and the Netherlands two each, and Luxembourg one vote, for a total of seventeen votes, with twelve of the seventeen votes needed for passage. Now suppose that in our hypothetical example
we suggest mimicking this with the United States getting three votes while each of its two smaller neighbors gets one vote. With this total of five votes we could also suggest using majority rule (three or more out of five votes) for passage and argue that it is not unreasonable for the United States to have three times as much “power” as either Canada or Mexico. In this situation it is certainly unlikely that either Canada or Mexico would be willing to go along with the previously suggested intuition aligning “three times as many votes” with “three times as much power.”

In the hypothetical example above, Canada and Mexico have no “power” (although they have votes). So what is this aspect of power that they are completely without? As an answer, “control over outcomes” suggests itself, and, indeed, much of the present chapter is devoted to quantitative measures of power that directly incorporate this control-over-outcomes aspect of power. (It also turns out—and we’ll discuss this in more detail later—that these quantitative measures of power indicate that Luxembourg fared no better in the original European Economic Community of 1958 than would Canada and Mexico in our hypothetical example.)

In Section 4.2 we consider the most well-known cardinal notion of power: the Shapley–Shubik index. This notion of power applies to any yes–no voting system (and not to just weighted voting systems). The mathematical preliminaries involved here include the “multiplication principle” and its corollary giving the number of distinct arrangements of \( n \) objects. In Section 4.3, we calculate the Shapley–Shubik indices for the European Economic Community, and we use later developments in this voting system to illustrate a phenomenon known as the “paradox of new members” (where one’s power is actually increased in a situation where it appears to have been diluted). In Section 4.4, we consider “voting blocs” and we prove a general theorem from Straffin (1980) that allows one to calculate the Shapley–Shubik index of such a bloc in a fairly trivial way.

Section 4.5 contains the second most well-known cardinal notion of power: the Banzhaf index. In Section 4.6, we introduce two methods, dating back to Allingham (1975) and Dahl (1957), for calculating the Banzhaf index, and we illustrate these methods using both the European Economic Community and a new paradox of Felsenthal and Machover (1994) wherein a voter’s power, as measured by the Banzhaf
4.2. The Shapley-Shubik Index of Power

index, increases after giving away a vote. (A paradox, due to William Zwicker, that applies to the Shapley-Shubik index but not the Banzhaf index is given in the exercises.)

In Section 4.7 we offer a precise mathematical definition that is intended to capture the intuitive notion of what it means to say that two voters have incomparable power. This definition provides us with what is called an ordinal notion of power, and we show that it is closely linked to the idea of “swap robustness” from the last chapter. The cognoscenti will recognize this as leading to the well-known “desirability relation on individuals” (developed fully in Chapter 9).

Before continuing, we add one convention that will be in place throughout this chapter (and Chapter 9):

**CONVENTION.** Whenever we say “voting system” we mean “monotone voting system in which the grand coalition (the one to which all the voters belong) is winning, and the empty coalition (the one to which none of the voters belongs) is losing.”

With this convention at hand, we can now turn to our discussion of power.

4.2 THE SHAPLEY-SHUBIK INDEX OF POWER

We begin with some mathematical preliminaries. Suppose we have \( n \) people \( p_1, p_2, \ldots, p_n \) where \( n \) is some positive integer. In how many different ways (i.e., orders) can we arrange them? We check it for some small values of \( n \) in Figure 1.

\[
\begin{align*}
 n = 1: & \text{ clearly only one way } \quad p_1 \\
 n = 2: & \text{ two ways } \quad p_2p_1 \text{ and } p_1p_2 \\
 n = 3: & \text{ six ways } \quad p_3p_2p_1; p_2p_3p_1; p_2p_1p_3 \\
 & \text{ and } \quad p_3p_1p_2; p_1p_3p_2; p_1p_2p_3
\end{align*}
\]

**Figure 1**
4.2. The

\[
\begin{align*}
\text{PROPOSITION 1 (The Multiplication Principle).} & \text{ Suppose we are} \\
& \text{considering objects each of which can be built in two steps. Suppose}
\end{align*}
\]
there are exactly \( f \) (for "first") ways to do the first step and exactly \( s \) (for "second") ways to do the second step. Then the number of such objects (that can be built altogether) is the product \( f \times s \). (We are assuming that different construction scenarios produce different objects.)

**PROOF.** Consider the tree in Figure 4, where the nodes labeled 1, 2, \ldots, \( f \) on the first level represent the \( f \) ways to do the first step in the construction process, and, for each of these, the nodes labeled 1, 2, \ldots, \( s \) represent the \( s \) ways to do the second step.

Notice that each terminal node corresponds to a two-step construction scenario. Moreover, the number of terminal nodes is clearly \( f \times s \) since we have \( f \) "clumps" (one for each node on level 1) and each "clump" is of size \( s \). This completes the proof.

Suppose now that we are building objects by a three-step process where there are \( k_1 \) ways to do the first step, \( k_2 \) ways to do the second, and \( k_3 \) ways to do the third step. How many such objects can be constructed? The answer, it turns out, can be derived from Proposition 1 because we can regard this three-step process as taking place in two "new steps" as follows:

1. New step one: same as old step one.
2. New step two: do the old step two and then the old step three.

Notice that Proposition 1 tells us there are \( k_2 \times k_3 \) new step twos. Since we know there are \( k_1 \) new step ones, we can apply Proposition 1
again to conclude that the number of objects built by our new two-step process (equivalently, by our old three-step process) is given by:

\[ k_1 \times (k_2 \times k_3) = k_1 \times k_2 \times k_3. \]

One could also derive this by looking at a tree with three levels, and the general result—stated below as Proposition 2—is usually derived via a proof technique known as mathematical induction. We'll content ourselves here, however, with simply recording the general result and the corollary.

**Proposition 2 (The General Multiplication Principle).** Suppose we are considering objects all of which can be built in \( n \) steps. Suppose there are exactly \( k_1 \) ways to do the first step, \( k_2 \) ways to do the second step, and so on up to \( k_n \) ways to do the \( n^{th} \) step. Then the number of such objects (that can be built altogether) is

\[ k_1 \times k_2 \times \ldots \times k_n, \]

assuming that different construction scenarios produce different objects.

As an application of Proposition 2, suppose we have \( n \) people and the objects we are building are arrangements (i.e., orders) of the people. Each ordering can be described as taking place in \( n \) steps as follows:

Step 1: Choose one person (from the \( n \)) to be first.

Step 2: Choose one person (from the remaining \( n - 1 \)) to be second.

\[ \vdots \]

Step \( n - 1 \): Choose one person (from the remaining 2) to be \( n - 1^{st} \).

Step \( n \): Choose the only remaining person to be last.

Clearly there are \( n \) ways to do step 1, \( n - 1 \) ways to do step 2, \( n - 2 \) ways to do step 3 and so on down to 2 ways to do step \( n - 1 \) and 1 way to do step \( n \). Thus, an immediate corollary of Proposition 2 is the following:
4.2. The Shapley-Shubik Index of Power

**COROLLARY.** The number of different ways that \( n \) people can be arranged is

\[ n \times (n - 1) \times (n - 2) \times \cdots \times (3) \times (2) \times (1), \]

which is, of course, just \( n! \) (factorial, not surprise).

One final idea—that of a “pivotal player”—is needed before we can present the formal definition of the Shapley–Shubik index. Suppose, for example, that we have a yes–no voting system with seven players: \( p_1, p_2, p_3, p_4, p_5, p_6, p_7 \). Fix one of the 7! orderings; for example, let’s consider:

\[ p_3, p_5, p_1, p_6, p_7, p_4, p_2. \]

We want to identify one of the players as being “pivotal” for this ordering. To explain this idea, we picture a larger and larger coalition being formed as we move from left to right. That is, we first have \( p_3 \) alone, then \( p_5 \) joins to give us the two-member coalition \( p_3, p_5 \). Then \( p_1 \) joins, yielding the three-member coalition \( p_3, p_5, p_1 \). And so on. The pivotal person for this ordering is the one whose joining converts this growing coalition from a non-winning one to a winning one. Since the empty coalition is losing and the grand coalition is winning (by our convention in Section 4.1), it is easy to see that some voter must be pivotal.

**Example:**

Suppose \( X = \{p_1, \ldots, p_7\} \) and each player has one vote except \( p_4 \) who has three. Suppose five votes are needed for passage. Consider the ordering: \( p_7, p_3, p_4, p_2, p_1, p_6 \). Then, since \( \{p_7, p_3, p_4\} \) is not a winning coalition, but \( \{p_7, p_3, p_5, p_4\} \) is a winning coalition, we have that the pivotal player for this ordering is \( p_4 \).

The Shapley–Shubik index of a player \( p \) is the number between zero and one that represents the fraction of orderings for which \( p \) is the pivotal player. Thus, being pivotal for lots of different orderings corresponds to having a lot of power according to this particular way of measuring power. More formally, the definition runs as follows.
4.2. The Shapley-Shubik Index of Power

**COROLLARY.** The number of different ways that \( n \) people can be arranged is

\[
n \times (n - 1) \times (n - 2) \times \cdots \times (3) \times (2) \times (1),
\]

which is, of course, just \( n! \) (factorial, not surprise).

One final idea—that of a “pivotal player”—is needed before we can present the formal definition of the Shapley–Shubik index. Suppose, for example, that we have a yes-no voting system with seven players: \( p_1, p_2, p_3, p_4, p_5, p_6, p_7 \). Fix one of the \( 7! \) orderings; for example, let’s consider:

\[
p_5, p_2, p_3, p_6, p_7, p_4, p_1.
\]

We want to identify one of the players as being “pivotal” for this ordering. To explain this idea, we picture a larger and larger coalition being formed as we move from left to right. That is, we first have \( p_5 \) alone, then \( p_5 \) joins to give us the two-member coalition \( p_3, p_5 \). Then \( p_1 \) joins, yielding the three-member coalition \( p_3, p_5, p_1 \). And so on. The pivotal person for this ordering is the one whose joining converts this growing coalition from a non-winning one to a winning one. Since the empty coalition is losing and the grand coalition is winning (by our convention in Section 4.1), it is easy to see that some voter must be pivotal.

**Example:**

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The Shapley–Shubik index of a player \( p \) is the number between zero and one that represents the fraction of orderings for which \( p \) is the pivotal player. Thus, being pivotal for lots of different orderings corresponds to having a lot of power according to this particular way of measuring power. More formally, the definition runs as follows.
4.3 Political Power

**Definition.** Suppose \( p \) is a voter in a yes–no voting system and let \( X \) be the set of all voters. Then the *Shapley–Shubik index* of \( p \), denoted here by \( \text{SSI}(p) \), is the number given by:

\[
\text{SSI}(p) = \frac{\text{the number of orderings of } X \text{ for which } p \text{ is pivotal}}{\text{the total number of possible orderings of the set } X}.
\]

Note the following:

1. The denominator in \( \text{SSI}(p) \) is just \( n! \) if there are \( n \) voters.
2. For every voter \( p \) we have \( 0 \leq \text{SSI}(p) \leq 1 \).
3. If the voters are \( p_1, \ldots, p_n \), then \( \text{SSI}(p_1) + \ldots + \text{SSI}(p_n) = 1 \).

Intuitively, think of \( \text{SSI}(p) \) as the “fraction of power” that \( p \) has. The following easy example is taken from Brams (1975); it is somewhat striking.

**Example:**

Suppose we have a three-person weighted voting system in which \( p_1 \) has fifty votes, \( p_2 \) has forty-nine votes, and \( p_3 \) has one vote. Assume fifty-one votes are needed for passage. The six possible orderings (\( 3! = 3 \times 2 \times 1 = 6 \)) are listed below, and the pivotal player for each has been circled.

\[
\begin{array}{ccc}
p_1 & p_2 & p_3 \\
p_1 & p_3 & p_2 \\
p_2 & p_1 & p_3 \\
p_2 & p_3 & p_1 \\
p_3 & p_1 & p_2 \\
p_3 & p_2 & p_1 \\
\end{array}
\]

Since \( p_1 \) is pivotal in four of the orderings, \( \text{SSI}(p_1) = \frac{4}{6} = \frac{2}{3} \).

Since \( p_2 \) is pivotal in one of the orderings, \( \text{SSI}(p_2) = \frac{1}{6} \).

Since \( p_3 \) is pivotal in one of the orderings, \( \text{SSI}(p_3) = \frac{1}{6} \).
Notice that although \( p_2 \) has forty-nine times as many votes as \( p_1 \), they each have the same fraction of power (at least according to this particular way of measuring power).

### 4.3 Calculations for the European Economic Community

We now return to the European Economic Community as set up in 1958 and calculate the Shapley–Shubik index for the member countries. Recall that France, Germany, and Italy had four votes, Belgium and the Netherlands had two votes, and Luxembourg had one vote. Passage required at least twelve of the seventeen votes.

Let's begin by calculating calculate SSI(France). We'll need to determine how many of the \( 6! = 720 \) different orderings of the six countries have France as the pivotal player. Because 720 is a fairly large number, we will want to get things organized in such a way that we can avoid looking at the 720 orderings one at a time.

Notice first that France is pivotal for an ordering precisely when the number of votes held by the countries to the left of it is either eight, nine, ten, or eleven. (If the number were seven or less, then the addition of France's four votes would yield a total of at most eleven, and thus not make it a winning coalition. If the number were twelve or more, it would be a winning coalition without the addition of France.) We'll handle these four cases separately, and then just add together the number of orderings from each case in which France is pivotal to get the desired final result.

**Case 1: Exactly Eight Votes Precede France**

There are three ways to total eight with the remaining numbers. We'll handle each of these as a subcase.

**1.1: France is Preceded by Germany, Belgium, and the Netherlands (with Votes 4, 2, and 2)**

In this subcase, the three countries preceding France can be ordered in \( 3! = 6 \) ways, and for each of these six, the two countries following France (Italy and Luxembourg in this case) can be ordered...
4.3. Calculations for the European Economic Community

Case 3: Exactly Ten Votes Precede France

3.1: France is Preceded by Germany, Italy, and Belgium (with Votes 4, 4, and 2)

The number of orderings turns out to be $3! \times 2! = 6 \times 2 = 12$.

3.2: France is Preceded by Germany, Italy, and the Netherlands (with Votes 4, 4, and 2)

Exactly as in 3.1, the number here is 12.

Hence, in case 3 we have a total of 24 distinct orderings in which France is pivotal.

Case 4: Exactly Eleven Votes Precede France

4.1: France is Preceded by Germany, Italy, Belgium, and Luxembourg (with Votes 4, 4, 2, and 1)

The number of orderings turns out to be $4! \times 1! = 24 \times 1 = 24$.

4.2: France is Preceded by Germany, Italy, the Netherlands, and Luxembourg (with Votes 4, 4, 2, and 1)

Exactly as in 4.1, the number here is 24.

Hence, in case 4 we have a total of 48 distinct orderings in which France is pivotal.

Finally, to calculate the Shapley–Shubik index of France, we simply add up the number of orderings from the above four cases (giving us the number of orderings for which France is pivotal), and divide by the number of distinct ways of ordering six countries (which is $6! = 720$). Thus,

$$SSI(\text{France}) = \frac{36 + 60 + 24 + 48}{720} = \frac{168}{720} = \frac{14}{60} \approx 23.3\%$$

Germany and Italy also have a Shapley–Shubik index of $14/60$ since, like France, they have four votes. It turns out that the Netherlands and Belgium both have a Shapley–Shubik index of $9/60$, although we'll leave this as an exercise (which can be done in two different ways).
in \(2! = 2\) ways. Thus we have \(6 \times 2 = 12\) distinct orderings in this subcase. (Equivalently, the number of orderings in this subcase—by Proposition 2—is \(3 \times 2 \times 1 \times 1 \times 2 \times 1 = 12\).)

1.2: France is Preceded by Italy, Belgium, and the Netherlands (with Votes 4, 2, and 2)

This case is exactly as 1.1, since both Germany and Italy have four votes.

1.3: France is Preceded by Germany and Italy (with Votes 4 and 4)

In this subcase, the two countries preceding France can be ordered in \(2! = 2\) ways, and for each of these two, the three countries following France (Belgium, the Netherlands, and Luxembourg in this case) can be ordered in \(3! = 6\) ways. Thus we have \(2 \times 6 = 12\) distinct orderings in this subcase, also.

Hence, in case 1 we have a total of 36 distinct orderings in which France is pivotal. For the next three cases (and their subcases), we'll leave the calculations to the reader and just record the results.

Case 2: Exactly Nine Votes Precede France

2.1: France is Preceded by Germany, Belgium, the Netherlands, and Luxembourg (with Votes 4, 2, 2, and 1)

The number of orderings here turns out to be \(4! \times 1! = 24 \times 1 = 24\).

2.2: France is Preceded by Italy, Belgium, the Netherlands, and Luxembourg (with Votes 4, 2, 2, and 1)

As in 2.1, the number of orderings here is 24.

2.3: France is Preceded by Germany, Italy, and Luxembourg (with Votes 4, 4, and 1)

The number of orderings turns out to be \(3! \times 2! = 6 \times 2 = 12\).

Hence, in case 2 we have a total of 60 distinct orderings in which France is pivotal.
at the end of the chapter. Another exercise is to show that poor Luxembourg has a Shapley–Shubik index of zero! (Hint: in order for Luxembourg to be pivotal in an ordering, exactly how many votes would have to be represented by countries preceding Luxembourg in the ordering? What property of the numbers giving the votes for the other five countries makes this total impossible?) These results are summarized in the following chart:

<table>
<thead>
<tr>
<th>Country</th>
<th>Votes</th>
<th>Percentage of votes</th>
<th>SSI</th>
<th>Percentage of power</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>4</td>
<td>23.5</td>
<td>14/60</td>
<td>23.3</td>
</tr>
<tr>
<td>Germany</td>
<td>4</td>
<td>23.5</td>
<td>14/60</td>
<td>23.3</td>
</tr>
<tr>
<td>Italy</td>
<td>4</td>
<td>23.5</td>
<td>14/60</td>
<td>23.3</td>
</tr>
<tr>
<td>Belgium</td>
<td>2</td>
<td>11.8</td>
<td>9/60</td>
<td>15.0</td>
</tr>
<tr>
<td>Netherlands</td>
<td>2</td>
<td>11.8</td>
<td>9/60</td>
<td>15.0</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>1</td>
<td>5.9</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We conclude this section by using the European Economic Community to illustrate a well-known paradox that arises with cardinal notions of power such as those considered in the present chapter (and later in Chapter 9). The setting is as follows: Suppose we have a weighted voting body as set up among France, Germany, Italy, Belgium, the Netherlands, and Luxembourg in 1958. Suppose now that new members are added and given votes, but the percentage of votes needed for passage remains about the same. Intuitively, one would expect the "power" of the original players to become somewhat diluted, or, at worse, to stay the same. The rather striking fact that this need not be the case is known as the "Paradox of New Members." It is, in fact, precisely what occurred when the European Economic Community expanded in 1973.

Recall that in the original European Economic Community, France, Germany, and Italy each had four votes, Belgium and the Netherlands each had two votes, and Luxembourg had one, for a total of seventeen. Passage required twelve votes, which is 70.6 percent of the seventeen available votes. In 1973, the European Economic Community was expanded by the addition of England, Denmark, and Ireland. It was decided that England should have the same number of votes as France, Germany, and Italy, but that Denmark and Ireland should have more...
4.4. A Theorem on Voting Blocs

votes than the one held by Luxembourg and fewer than the two held by Belgium and the Netherlands. Thus, votes for the original members were scaled up by a factor of $2 \frac{1}{2}$, except for Luxembourg, which only had its total doubled. In summary then, the countries and votes stood as follows:

<table>
<thead>
<tr>
<th>Country</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>10</td>
</tr>
<tr>
<td>Belgium</td>
<td>5</td>
</tr>
<tr>
<td>England</td>
<td>10</td>
</tr>
<tr>
<td>Germany</td>
<td>10</td>
</tr>
<tr>
<td>Netherlands</td>
<td>5</td>
</tr>
<tr>
<td>Denmark</td>
<td>3</td>
</tr>
<tr>
<td>Italy</td>
<td>10</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>2</td>
</tr>
<tr>
<td>Ireland</td>
<td>3</td>
</tr>
</tbody>
</table>

The number of votes needed for passage was set at forty-one, which is 70.7 percent of the fifty-eight available votes.

The striking thing to notice is that Luxembourg's power—as measured by the Shapley–Shubik index—has increased. That is, while Luxembourg's Shapley–Shubik index had previously been zero, it is clearly greater than zero now since we can produce at least one ordering of the nine countries for which Luxembourg is pivotal. (The actual production of such an ordering is left as an exercise at the end of the chapter.) Notice also that this increase of power is occurring in spite of the fact that Luxembourg was treated worse than the other countries in the scaling-up process. For some even more striking instances of this paradox of new members phenomenon, see the exercises at the end of the chapter where, for example, it is pointed out that even if Luxembourg had been left with one vote, its power still would have increased.

4.4 A THEOREM ON VOTING BLOCS

This section considers a situation that reduces to a kind of weighted voting body that is sufficiently simple so that we can prove a general theorem, taken from Straffin (1980), that allows us to calculate the Shapley–Shubik indices of the players involved in an easy way. We begin with some notation and an example.

**NOTATION.** Suppose we have a weighted voting system with $n$ players $\rho_1, \ldots, \rho_n$ with weights $w_1, \ldots, w_n$ (so, $w_i$ is the weight of player $\rho_i$, $w_1$
of \( q_2, \) etc.) Suppose that \( q \) is the quota. Then all of this is denoted by:

\[
[q : w_1, w_2, \ldots, w_n].
\]

For example, the European Economic Community is the system:

\[
[12 : 4, 4, 4, 2, 2, 1]
\]

With this notation at hand, we now turn to an extended example of how the Shapley–Shubik index of a voting bloc can be calculated.

**Extended Example:**

Consider the United States Senate as a yes-no voting system with 100 voters, each of whom has one vote and with fifty-one votes needed for passage. (We ignore the vice president.) Thus, the Shapley–Shubik index of any one senator is 1/100, since they must all be the same and sum to one. But now suppose the twelve senators from the six New England states decide to vote together as a so-called voting bloc. Intuitively, the power of this bloc would seem to be greater than the sum of the powers of the individual senators. Our goal in this example is to treat the bloc as a single player with twelve votes, and to calculate the Shapley–Shubik index of this bloc. The result obtained should quantify the above intuition.

Consider, then, the weighted voting body \([51: 12, 1, 1, \ldots, 1]\) where there are eighty-eight ones (representing the senators from the forty-four non-New England states). The first thing to notice is that we really don’t have to consider all possible ways of ordering the eighty-nine players. That is, this collection of \(89!\) orderings is broken into 89 equal size “clumps” determined by the place occupied by the “12” in the string of eighty-eight ones. The eighty-nine “places” are pictured below.

\[
\begin{array}{cccccccc}
1 & 1 & \ldots & 1 & 1 & 1 \\
\text{place 1} & \text{place 2} & \text{place 88} & \text{place 89} \\
\end{array}
\]

(88 ones all together)
4.4. A Theorem on Voting Blocs

The different orderings within any one clump are arrived at simply by permuting the "ones" involved. In particular, then, the twelve-vote player is pivotal for one ordering in the clump only if it is pivotal for every ordering in the clump. Thus, to calculate the Shapley–Shubik index of the voting bloc, we must simply determine how many of the eighty-nine distinct orderings (one from each clump) have the "12" in a pivotal position.

In order for the "12" to be in the pivotal position, the number of ones preceding it must be at least thirty-nine. That is, if there were thirty-eight or fewer then the addition of the twelve-vote bloc would yield a coalition with fewer than the fifty-one votes needed to make it a winning coalition. On the other hand, if the number of ones preceding the "12" were more than fifty, then the coalition would be winning before the "12" joined. Thus, the orderings that have "12" in the pivotal position are the ones with an initial sequence of ones of length 39, 40, ..., 50. There are twelve numbers in this sequence. Hence, we can conclude the following:

\[ \text{SSI(New England bloc)} = \frac{12}{89}. \]

Thus, although the fraction of votes held by the New England bloc is only 12/100, the fraction of power (as measured by the Shapley–Shubik index) is 12/89.

The answer of 12/89 arrived at in the above example is easy to remember in terms of the parameters of the problem. That is, the numerator is just the size of the voting bloc, while the denominator is the number of distinct players when the bloc is considered to be a single player. The following theorem tells us that this is no coincidence.

**Theorem.** Suppose we have \( n \) players and that a single bloc of size \( b \) forms. Consider the resulting weighted voting body:

\[
[q : b, 1, 1, \ldots, 1] \quad \underbrace{\text{\(n-b\) of these}}_{\text{\(n-b\) of these}}
\]

Assume \( b - 1 \leq q - 1 \leq n - b \). Then the Shapley–Shubik index of the bloc is given by:

\[ \text{SSI}(\text{bloc}) = \frac{b}{n - b + 1}. \]
PROOF. The argument is just a general version of what we did before. Notice first that $n - b + 1$ is just the number of distinct orderings. (Recall that we are not distinguishing between two orderings in which the ones have been rearranged.) Thus, we have $n - b$ ones and the number of places the $b$ can be inserted is just one more than this. (In the example above, we had eighty-eight ones and eighty-nine places to insert the 12 bloc.)

The $b$ bloc will be pivotal precisely when the initial sequence of ones is of length at least $q - b$ (since $q - b + b$ is just barely the quota $q$), but not more than $q - 1$ (or else the quota is achieved without $b$). Hence, the $b$ bloc is pivotal when the initial sequence of ones is any of the following lengths:

$q - 1, q - 2, \ldots, q - b$.

Notice that since $q - 1 \leq n - b$ and $n - b$ is the number of ones available, we can construct all of these sequences—even the one with the initial segment requiring $q - 1$ ones. Clearly, there are exactly $b$ numbers in the above list. Thus, the Shapley–Shubik index of the bloc of size $b$ is given by:

$$SSI(b \text{ bloc}) = \frac{\text{number of orders in which } b \text{ is pivotal}}{\text{total number of distinct orderings}} = \frac{b}{n - b + 1}.$$  

4.5 THE BANZHAF INDEX OF POWER

A measure of power that is similar to (but not the same as) the Shapley–Shubik index is the so-called Banzhaf index of a player. This power index was introduced by the attorney John F. Banzhaf III in connection with a law suit involving the county board of Nassau County, New York in the 1960s (see Banzhaf, 1965). The definition takes place via the intermediate notion of what we shall call the “total Banzhaf power” of a player. The definition follows.

DEFINITION. Suppose that $p$ is a voter in a yes–no voting system. Then the total Banzhaf power of $p$, denoted here by $TBP(p)$, is the number of coalitions $C$ satisfying the following three conditions:
4.5. The Banzhaf Index of Power

1. \( p \) is a member of \( C \).
2. \( C \) is a winning coalition.
3. If \( p \) is deleted from \( C \), the resulting coalition is not a winning one.

If \( C \) is a winning coalition, but the coalition resulting from \( p \)'s deletion from \( C \) is not, then we say that “\( p \)'s defection from \( C \) is critical.” Notice that \( \text{TBP}(p) \) is an integer (whole number) as opposed to a fraction between zero and one. To get such a corresponding fraction, we do the following (which is called “normalizing”).

**Definition.** Suppose that \( p_1 \) is a player in a yes–no voting system and that the other players are denoted by \( p_2, p_3, \ldots, p_n \). Then the Banzhaf index of \( p_1 \), denoted here by \( \text{BI}(p_1) \), is the number given by

\[
\text{BI}(p_1) = \frac{\text{TBP}(p_1)}{\text{TBP}(p_2) + \cdots + \text{TBP}(p_n)}.
\]

Notice that \( 0 \leq \text{BI}(p) \leq 1 \) and that if we add up the Banzhaf indices of all \( n \) players, we get the number 1.

**Example:**

Let's again use the example where the voters are \( p_1, p_2, \) and \( p_3 \); and \( p_1 \) has fifty votes, \( p_2 \) has forty-nine votes, \( p_3 \) has one vote; and fifty-one votes are needed for passage. We will calculate \( \text{TBP} \) and \( \text{BI} \) for each of the three players. Recall that the winning coalitions are

\[
C_1 = \{p_1, p_2, p_3\},
\]
\[
C_2 = \{p_1, p_2\},
\]
\[
C_3 = \{p_1, p_3\}.
\]

For \( \text{TBP}(p_1) \), we see that \( p_1 \) is in each of the three winning coalitions and his defection from each is critical. On the other hand, neither \( p_2 \)'s nor \( p_1 \)'s defection from \( C_1 \) is critical, but \( p_2 \)'s is from \( C_2 \) and \( p_3 \)'s is from \( C_3 \). Thus:

\[
\text{TBP}(p_1) = 3 \quad \text{TBP}(p_2) = 1 \quad \text{TBP}(p_3) = 1
\]
and, thus,

\[
\begin{align*}
\text{BI}(p_1) &= \frac{3}{(3 + 1 + 1)} = \frac{3}{5} \\
\text{BI}(p_2) &= \frac{1}{(3 + 1 + 1)} = \frac{1}{5} \\
\text{BI}(p_3) &= \frac{1}{(3 + 1 + 1)} = \frac{1}{5}.
\end{align*}
\]

Recall that for the same example we had \( \text{SSI}(p_1) = \frac{2}{3} \), \( \text{SSI}(p_2) = \frac{1}{6} \), and \( \text{SSI}(p_3) = \frac{1}{6} \).

### 4.6 Two Methods of Computing Banzhaf Power

This section presents two new procedures for calculating total Banzhaf power. Both procedures begin with a very simple chart that has the winning coalitions enumerated in a vertical list down the left side of the page, and the individual voters enumerated in a horizontal list across the top. For example, if the yes–no voting system is the original European Economic Community, the chart (with “T” for “France” etc.) will have:

\[
\begin{array}{ccccccc}
F & G & I & B & N & L \\
\end{array}
\]

across the top. Down the left side it will have the fourteen winning coalitions which turn out to be (displayed horizontally at the moment for typographical reasons):

- FGIL, FGBNL, FIBNL, GIBNL
- FGIB, FGIN
- FGIBL, FGINL
- FGIBN
- FGIBNL

Notice the order in which we have chosen to list the winning coalitions: the first four are precisely the ones with weight 12, the next four are the ones with weight 13, then the two with weight 14, the two
with weight 15, the one with weight 16, and the one with weight 17. If the voting system is weighted, this is a nice way to ensure that no winning coalitions have been missed. In what follows, we shall need the observation that there are fourteen winning coalitions in all.

We now present and illustrate the two procedures for calculating total Banzhaf power. Notice that “critical defection” is not mentioned in either procedure.

**PROCEDURE 1.** Assign each voter (country) a “plus one” for each winning coalition of which it is a member, and assign it a “minus one” for each winning coalition of which it is not a member. The sum of these “plus and minus ones” turns out to be the total Banzhaf power of the voter. (The reader wishing to get ahead of us should stop here and contemplate why this is so.) Continuing with the European Economic Community as an example, we have:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>G</th>
<th>I</th>
<th>B</th>
<th>N</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGI</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>FGBN</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>FBIN</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>GIBN</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>FGIL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>FGBNL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>FIBNL</td>
<td>1</td>
<td>−1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>GIBNL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>FGIB</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>FGIN</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>1</td>
<td>−1</td>
</tr>
<tr>
<td>FGIBL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>FGINL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>FGIBN</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
</tr>
<tr>
<td>FGIBNL</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>TBP(sum)</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

**PROCEDURE 2.** Assign each voter (country) a “plus two” for each winning coalition in which it appears (and assign it nothing for those in which it does not appear). Subtract the total number of winning coalitions from this sum. The answer turns out to be the total Banzhaf power of the voter.
Continuing with the European Economic Community as an example, we have:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>G</th>
<th>I</th>
<th>B</th>
<th>N</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGI</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FGBN</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>FIBN</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>GIBN</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>FGBNL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FIBNL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>GIBNL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIB</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIN</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIBL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGINL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIBN</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>FGIBNL</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(sum) 24 24 24 20 20 14

Minus number of winning coalitions -14 -14 -14 -14 -14 -14

TBP 10 10 10 6 6 0

The following chart summarizes the Banzhaf indices (arrived at by dividing each country’s total Banzhaf power by 10 + 10 + 10 + 6 + 6 + 0 = 42). This is analogous to what we did for the Shapley–Shubik indices in Section 4.3.

<table>
<thead>
<tr>
<th>Country</th>
<th>Votes</th>
<th>Percentage of votes</th>
<th>BI</th>
<th>Percentage of power</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>4</td>
<td>23.5</td>
<td>5/21</td>
<td>23.8</td>
</tr>
<tr>
<td>Germany</td>
<td>4</td>
<td>23.5</td>
<td>5/21</td>
<td>23.8</td>
</tr>
<tr>
<td>Italy</td>
<td>4</td>
<td>23.5</td>
<td>5/21</td>
<td>23.8</td>
</tr>
<tr>
<td>Belgium</td>
<td>2</td>
<td>11.8</td>
<td>3/21</td>
<td>14.3</td>
</tr>
<tr>
<td>Netherlands</td>
<td>2</td>
<td>11.8</td>
<td>3/21</td>
<td>14.3</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>1</td>
<td>5.9</td>
<td>0/5</td>
<td>0</td>
</tr>
</tbody>
</table>
4.6. Two Methods of Computing Banzhaf Power

Why is it that these two procedures give us the number of critical defections for each voter? Let's begin with the following easy observation: Procedure 2 yields the same numbers as does Procedure 1. That is, in going from Procedure 1 to Procedure 2, all the “minus ones” became “zeros” and all the “plus ones” became “twos.” Hence, the sum for each voter increased by one for each winning coalition. Thus, when we subtracted off the number of winning coalitions, the result from Procedure 2 became the same as the result from Procedure 1.

So we need only explain why Procedure 1 gives us the number of critical defections that each voter has. (Recall that we are considering only monotone voting systems.) The key to understanding what is happening in Procedure 1 is to have at hand a particularly revealing enumeration of the winning coalitions. Such a revealing enumeration arises from focusing on a single voter \( p \), with different voters in the role of \( p \) giving different enumerations. To illustrate such an enumeration, let's let the fixed voter \( p \) be the country Belgium in the European Economic Community. The list of winning coalitions corresponding to the fixed voter \( p \) will be made up of three “blocks” of coalitions:

Block 1: Those winning coalitions that do not contain \( p \).

Block 2: The coalitions in Block 1 with \( p \) added to them.

Block 3: The rest of the winning coalitions.

For example, with Belgium playing the role of the fixed voter \( p \), we would have the fourteen winning coalitions in the European Economic Community listed in the following order:

Block 1:  FGI  
            FGIL  
            FGIN  
            FGINL

Block 2:  FGIB  
            FGILB  
            FGINB  
            FGINLB
4. Political Power

Block 3:  
FGNB  
FINB  
GINB  
FGNLB  
FINLB  
GINLB

There are several things to notice about the blocks. First, the coalitions in Block 2 are all winning because those in Block 1 are winning and we are only considering monotone voting systems. Second, there are exactly as many coalitions in Block 2 as in Block 1, because if \(X\) and \(Y\) are two distinct winning coalitions in Block 1, and thus neither contains \(p\), then adding \(p\) to each of \(X\) and \(Y\) will again result in distinct coalitions in Block 2. Moreover, every coalition in Block 2 arises from one in Block 1 in this way. Third, every coalition in Block 3 contains \(p\), since all those not containing \(p\) were listed in Block 1.

Finally, and perhaps most importantly, is the observation that \(p\)'s defection from a winning coalition is critical precisely for the coalitions in Block 3. That is, \(p\) does not even belong to the coalitions in Block 1, and \(p\)'s defection from any coalition in Block 2 gives the corresponding winning coalition in Block 1, and thus is not critical. However, if \(p\)'s defection form a coalition \(X\) in Block 3 were to yield a coalition \(Y\) that is winning, then \(Y\) would have occurred in Block 1, and so \(X\) would have occurred in Block 2 instead of Block 3.

The reason Procedure 1 works is now clear: The minus ones in Block 1 are exactly offset by the plus ones in Block 2, thus leaving a plus one contribution for each coalition in Block 3 and these are precisely the ones for which \(p\)'s defection is critical.

Other consequences also follow. For example, a monotone game with exactly seventy-one winning coalitions has no dummies (Exercise 20 asks why). Notice that the listing of winning coalitions corresponding to the fixed voter \(p\) is used only to understand why the procedures work—such listings need not be constructed to actually calculate Banzhaf power using either Procedure 1 or Procedure 2.

Power indices tend to have some paradoxical aspects. For example, Felsenthal and Machover (1994) recently noticed the following paradoxical result for the Banzhaf index. Consider the weighted voting
4.7. Ordinal Power: Incomparability

system

\[ [8 : 5, 3, 1, 1, 1]. \]

The Banzhaf indicies of the voters turn out to be \( \frac{8}{19}, \frac{7}{19}, \frac{1}{19}, \frac{1}{19}, \) and \( \frac{1}{19} \) (Exercise 21). Now suppose that the voter with weight 5 gives one of his “votes” to the voter with weight 4. This results in the weighted system

\[ [8 : 4, 4, 1, 1, 1]. \]

It now turns out (Exercise 21 again) that the first voter has Banzhaf index \( \frac{1}{2} \). But \( \frac{1}{2} \) is greater than \( \frac{9}{19} \) (surprise, not factorial). Hence, by giving away a single vote to a single player (and no other changes being made), a player has increased his power as measured by the Banzhaf index. (Part of what is going here is that the transfer of a vote from the first player to the second makes each of the last three players a dummy. Hence, the first two players together share a larger fraction of the power than they previously did, and—as one would expect—the second player gains more than the first. The trade-off is that the first player is gaining more from the gain caused by the effective demise of the last three voters than he is losing from the transfer of one vote from himself to the second voter.) More on this paradox is found in Exercise 22.

It turns out (as pointed out by Felsenthal and Machover) that the Shapley–Shubik index is not vulnerable to this particular type of paradox. But the Shapley–Shubik index is not immune to such quirks: Exercise 23 presents a paradoxical aspect (due to William Zwicker) of the Shapley–Shubik index that is not shared by the Banzhaf index.

4.7 ORDINAL POWER: INCOMPARABILITY

We will continue to assume throughout this section that “yes–no voting system” means “monotone yes–no voting system.” Thus, winning coalitions remain winning if new voters join them.

Suppose we have a yes–no voting system (and, again, not necessarily a weighted one) and two voters whom we shall call \( x \) and \( y \). Our
starting point will be an attempt to formalize (that is, to give a rigorous mathematical definition for) the intuitive notion that underlies expressions such as the following:

"x and y have equal power"
"x and y have the same amount of influence"
"x and y are equally desirable in terms of the formation of a winning coalition"

The third phrase is most suggestive of where we are heading and, in fact, the thing we are leading up to—although we won’t completely arrive there until Section 9.5—is widely referred to as the “desirability relation on individuals” (although we could equally well call it the “power ordering on individuals” or the “influence ordering on individuals”). We shall begin with an attempt to formalize the notion of x and y having “equal influence” or being “equally desirable.”

If we think of the desirability of x and of y to a coalition Z, then there are four types of coalitions to consider:

1. x and y both belong to Z.
2. x belongs to Z but y does not.
3. y belongs to Z but x does not.
4. Neither x nor y belongs to Z.

If x and y are equally desirable (to the voters in Z, who want the coalition Z to be a winning one), then for each of the four situations described above, we have a statement that should be true:

1. If Z is a winning coalition, then x’s defection from Z should render it losing if and only if y’s defection from Z renders it losing.
2. If x leaves Z and y joins Z, then Z should go neither from being winning to being losing nor from being losing to being winning.
3. If y leaves Z and x joins Z, then Z should go neither from being winning to being losing nor from being losing to being winning.
4. $x$’s joining $Z$ makes $Z$ winning if and only if $y$’s joining $Z$ makes $Z$ winning.

In fact, it turns out that condition 4 is strong enough to imply the other three (see Exercises 25 and 26). This leads to the following definition:

**DEFINITION.** Suppose $x$ and $y$ are two voters in a yes–no voting system. Then we shall say that $x$ and $y$ are *equally desirable* (or, the desirability of $x$ and $y$ is equal, or the same), denoted $x \approx y$, if and only if the following holds:

For every coalition $Z$ containing neither $x$ nor $y$, the result of $x$ joining $Z$ is a winning coalition if and only if the result of $y$ joining $Z$ is a winning coalition.

For brevity, we shall sometimes just say: "$x$ and $y$ are equivalent" when $x \approx y$.

**Example:**

Consider again the weighted voting system with three players $a$, $b$, and $c$ who have weights 1, 49, and 50, respectively, and with quota $q = 51$. Then the winning coalitions are $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. Notice that $a \approx b$: the only coalitions containing neither $a$ nor $b$ are the empty coalition (call it $Z_1$) and the coalition consisting of $c$ alone (call it $Z_2$). The result of $a$ joining $Z_1$ is the same as the result of $b$ joining $Z_1$ (a losing coalition) and the result of a joining $Z_2$ is the same as the result of $b$ joining $Z_2$ (a winning coalition). On the other hand, $a$ and $c$ are not equivalent, since neither belongs to $Z = \{b\}$, but $a$ joining $Z$ yields $\{a, b\}$ which is losing with 50 votes, while $c$ joining $Z$ yields $\{b, c\}$ which is winning with 51 votes.

This example shows that in a weighted voting system, two voters with very different weights can be equivalent and, thus (intuitively) have the same “power” or “influence.”

The relation of “equal desirability” defined above will be further explored in **Section 9.5**. For now, however, we turn our attention to the question of when two voters not only fail to have equal influence,
but when it makes sense to say that their influence is “incomparable.” What should this mean? Mimicking what we did for the notion of equal desirability, let’s say that \( x \) and \( y \) are incomparable if one coalition \( Z \) desires \( x \) more than \( y \), and another coalition \( Z' \) desires \( y \) more than \( x \). Formalizing this yields:

**DEFINITION.** For two voters \( x \) and \( y \) in a yes–no voting system, we say that the *desirability of \( x \) and \( y \) is incomparable*, denoted

\[
x \parallel y,
\]

if and only if there are coalitions \( Z \) and \( Z' \), neither of which contains \( x \) or \( y \), such that the following hold:

1. the result of \( x \) joining \( Z \) is a winning coalition, but the result of \( y \) joining \( Z \) is a losing coalition, and
2. the result of \( y \) joining \( Z' \) is a winning coalition, but the result of \( x \) joining \( Z' \) is a losing coalition.

For brevity, we shall sometimes just say “\( x \) and \( y \) are incomparable” when \( x \parallel y \).

**Example:**

In the U.S. federal system, let \( x \) be a member of the House and let \( y \) be a member of the Senate. Then \( x \parallel y \) (see Exercise 28). On the other hand if \( x \) is the vice president and \( y \) is a member of the Senate, then \( x \) and \( y \) are not incomparable (see Exercise 29).

The following proposition characterizes exactly which yes–no voting systems will have incomparable voters. Recall from **Section 3.2** that a yes–no voting system is swap robust if a one-for-one exchange of players between two winning coalitions always leaves at least one of the two coalitions winning.

**PROPOSITION.** For any yes–no voting system, the following are equivalent:

1. There exist voters \( x \) and \( y \) whose desirability is incomparable.
4.7. Ordinal Power: Incomparability

2. The system fails to be swap robust.

**Proof.** (1 implies 2): Assume that the desirability of $x$ and $y$ is incomparable, and let $Z$ and $Z'$ be coalitions such that:

- $Z$ with $x$ added is winning;
- $Z$ with $y$ added is losing;
- $Z'$ with $y$ added is winning; and
- $Z'$ with $x$ added is losing.

To see that the system is not swap robust, let $X$ be the result of adding $x$ to the coalition $Z$, and let $Y$ be the result of adding $y$ to the coalition $Z'$. Both $X$ and $Y$ are winning, but the one-for-one swap of $x$ for $y$ renders both coalitions losing.

(2 implies 1): Assume the system is not swap robust. Then we can choose winning coalitions $X$ and $Y$ with $x$ in $X$ but not in $Y$, and $y$ in $Y$ but not in $X$, such that both coalitions become losing if $x$ is swapped for $y$. Let $Z$ be the result of deleting $x$ from the coalition $X$, and let $Z'$ be the result of deleting $y$ from the coalition $Y$. Then

- $Z$ with $x$ added is $X$, and this is winning;
- $Z$ with $y$ added is losing;
- $Z'$ with $y$ added is $Y$, and this is winning; and
- $Z'$ with $x$ added is losing.

This shows that the desirability of $x$ and $y$ is incomparable and completes the proof.

**Corollary.** In a weighted voting system, there are never voters whose desirability is incomparable.

**Proof.** In Section 3.2 we showed that a weighted voting system is always swap robust.

The question of what one can say about voters $x$ and $y$ whose desirability is neither equal nor incomparable is taken up in Section 9.5, but, in the meantime, the reader can try Exercise 31.
4.8 CONCLUSIONS

We began this chapter with a hypothetical example illustrating that "fraction of votes" and "fraction of power" need not be the same. Turning to the most well-known quantitative measure of power, we introduced the Shapley–Shubik index (and the necessary mathematical preliminaries, including the multiplication principle that will be used in later chapters as well). As an application of the Shapley–Shubik index we considered the European Economic Community as a weighted voting body and calculated the relative power of each of the six countries involved. The paradox of new members was illustrated by the 1973 expansion of the Common Market and the observation that Luxembourg's "power" increased instead of being diluted as intuition would suggest. We also presented a general theorem that allows the Shapley–Shubik index of a voting bloc to be calculated in a fairly trivial way.

The second most well-known quantitative measure of power—the Banzhaf index—was defined in terms of critical defections from winning coalitions. We then presented a couple of new procedures that allow one to calculate total Banzhaf power by making a single "run" down the list of winning coalitions, and we illustrated this with the European Economic Community.

Finally, in Section 4.7, we considered ordinal notions of power and introduced formal definitions intended to capture the intuitive idea of comparing the extent to which two voters are desired by coalitions that wish to become, or to remain, winning. The focus here was on a notion of when two voters have "incomparable" power, and we proved that a yes–no voting system has such voters if and only if it fails to be swap robust (as defined in Chapter 3).

EXERCISES

1. List out the 24 orderings of $p_1, p_2, p_3, p_4$. Arrange them so that the first four orderings in your list arise from the first ordering presented in Figure 3 in Section 4.2 (i.e., $p_3p_2p_1$), the next four from the second ordering presented in Figure 3 in Section 4.2 (i.e., $p_2p_3p_1$), etc.
TICAL POWER

Illustrate the twenty-four possible orderings of \( p_1, p_2, p_3, p_4 \) by drawing a tree with a start node at the top, four nodes on level one corresponding to a choice of which \( p_i \) will go first in the ordering, three nodes immediately below each of these on the next level corresponding to a choice of which \( p_j \) then will go second, etc.

3. Suppose we want to form a large governmental committee by choosing one of the two senators from each of the fifty states. How many distinct such committees can be formed? (Hint: Step 1 is to choose one of the two senators from Maine; there are two ways to do this. Step 2 is to choose \ldots ) Comment on the size of this number.

4. Show that there are fewer than 362,881 different games of tic-tac-toe with “X” going first. Note that two games of tic-tac-toe are different if there is a number \( n \), necessary between one and nine, so that the symbol being played (i.e., “X” or “O”) at move \( n \) is placed in different squares in the two games.

5. Show that there are fewer than 20,000 three-letter words in the English language.

6. Suppose that \( x \) has five votes, \( y \) has three votes, \( z \) has three votes, and \( w \) has two votes. Assume that eight votes are needed for passage. Calculate \( SSI(x) \), \( SSI(y) \), \( SSI(z) \), and \( SSI(w) \). Show all your work.

7. Show that Luxembourg has a Shapley–Shubik index of zero.

8. Show that the Netherlands and Belgium both have a Shapley–Shubik index of \( 9/60 \) in two different ways:
   (a) By directly calculating it as we did for France.
   (b) By using the fact that the sum of the indices must be one.

9. In the calculation of the Shapley–Shubik index of France (within the European Economic Community), the details were increasingly omitted as we proceeded from case 1 through case 4. Redo case 4 in as much detail as was provided for case 1.

10. Consider the 1973 expansion of the European Economic Community as described in Section 4.3. Show that Luxembourg has positive Shapley–Shubik Index by producing an ordering of the countries involved for which Luxembourg is pivotal.

11. Show that even if Luxembourg had been left with just one vote in the expansion of 1973 (with everything else as it actually was in the expansion), the Shapley–Shubik index of Luxembourg still would have been nonzero.
12. In the 1973 expansion of the European Economic Community, the percentage of votes needed for passage rose the trivial amount from 70.6 percent to 70.7 percent. Suppose they had decided to require forty (instead of forty-one) votes for passage. Would a paradox of new members still have taken place?

13. Consider the weighted voting body \([5 : 2, 2, 1, 1, 1, 1, 1]\). Calculate \(SSI(x)\) where \(x\) is one of the people with two votes. (Hint: The theorem in Section 4.4 does not directly apply here, but the underlying ideas of that theorem are all that is needed. That is, hold the “2” under consideration out, and ask how many distinct orderings of the remaining five 1s and one 2 there are. For each ordering, where is the insertion of the “2” under consideration pivotal?)

14. Use the theorem from Section 4.4 (where it applies) to calculate the Shapley–Shubik index of the voting bloc in each of the following weighted voting bodies. If the theorem doesn’t apply, say why.
   (a) \([5 : 7, 1, 1, 1]\)
   (b) \([8 : 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]\)
   (c) \([6 : 3, 1, 1, 1, 1, 1, 1]\)

15. Consider a voting system for the six New England states where there are a total of seventeen votes and twelve or more are required for passage. Votes are distributed as follows:

   MA:4 ME:3 NH:2
   CT:4 RI:3 VT:1

   (a) Calculate \(SSI(MA)\).
   (b) Calculate \(SSI(ME)\).
   (c) Calculate \(SSI(NH)\).
   (d) Calculate \(SSI(VT)\).

16. Suppose that \(x\) has five votes, \(y\) has three votes, \(z\) has three votes, and \(w\) has two votes. Assume that eight votes are needed for passage.
   (a) Calculate TBP for each voter by directly using the definition of TBP in terms of critical defections. Show all your work.
   (b) Calculate TBP for each voter by using Procedure 1 from Section 4.6. (There are seven winning coalitions.)
   (c) Calculate TBP for each voter by using Procedure 2 from Section 4.6. (There are fourteen winning coalitions.)
Exercises


18. Consider the “minority veto system” where there are six voters: A, B, C, D, E, and F, two of whom (E and F) form a designated minority, and passage requires a total of at least four of the six votes and at least one of the minority votes. Calculate the Banzhaf index for each of the voters in this system. (There are 21 winning coalitions.)

19. Explain how we know, in a monotone yes–no voting system, that every voter belongs to at least half the winning coalitions. Explain also why a voter in such a system is a dummy if and only if he belongs to exactly half the winning coalitions.

20. Explain how we know that there are no dummies in a monotone yes–no voting system with exactly 71 winning coalitions.

21. Use Procedure 1 or 2 to verify the calculations in the Felsenthal–Machover example in Section 4.6.

22. (a) Discuss whether or not you find it paradoxical that in going from the system [8 : 5, 3, 1, 1, 1] to the system [8 : 4, 7, 0, 0, 0], the first player’s Banzhaf power increases.

   (b) Discuss whether or not you find it paradoxical that two different choices of weights, such as [8 : 4, 7, 0, 0, 0] and [8 : 4, 4, 1, 1, 1] give the same yes–no voting system.

   (c) What, if anything, do your responses to (a) and (b) say about the Felsenthal–Machover paradox?

23. The Shapley–Shubik index is not without paradoxical aspects. For example, the following was pointed out to us by William Zwicker. Suppose we have a bicameral yes–no voting system wherein an issue must win in both the House and the Senate in order to pass. (We are not assuming that the House and Senate necessarily use majority rule, but we are assuming they have no common members.) Suppose that both you and I belong to the House and that—when the House is considered as a yes–no voting system in its own right—I have three times as much “power” as you have. Then shouldn’t I still have three times as much power as you have when we consider the bicameral yes–no voting system? (This is a rhetorical question.)

   (a) Suppose that $X_1, \ldots, X_m$ are the winning coalitions in the House and $Y_1, \ldots, Y_n$ are the winning coalitions in the Senate. Assume that I belong to $t$ of the winning coalitions in the House and that you belong to $z$ of the winning coalitions in the House.
1. Show that there are $mn$ winning coalitions in the bicameral system.

2. Use Procedure 2 to show that my total Banzhaf power in the House is $2t - m$ and that yours is $2z - m$.

3. Use Procedure 2 to show that my total Banzhaf power in the bicameral system is $2tn - mn$ and that yours is $2zn - mn$.

4. Show that if I have $v$ times as much power (as measured by the Banzhaf index) in the House as you have, then I also have $v$ times as much power (as measured by the Banzhaf index) in the bicameral system as you have.

(b) Suppose the House consists of you, me, and Bill, and suppose there are two minimal winning coalitions in the House: me alone (as one), and you and Bill together (as the other). In the Senate, there are two people and each alone is a minimal winning coalition.

1. Show that in the House alone, I have four times as much power—as measured by the Shapley–Shubik index—as you have. (The values turn out to be $\frac{4}{6}$, $\frac{2}{6}$, $\frac{1}{6}$.)

2. Show that in the bicameral system, this is no longer true. (The values turn out to be $\frac{44}{120}$ for me and $\frac{14}{120}$ for you.)

24. Consider the yes–no voting system in which there are six voters: $a, b, c, d, e, f$. Suppose the winning coalitions are precisely the ones containing at least two of $a, b,$ and $c$ and at least two of $d, e,$ and $f$.
(a) Show that $a$ and $b$ are equally desirable (as defined in Section 4.7).

(b) Show that the desirability of $a$ and $d$ is incomparable (using the definition in Section 4.7).

25. Suppose that $x$ and $y$ are voters in a yes–no voting system and that $x \approx y$. Suppose that $Z'$ is a winning coalition to which both $x$ and $y$ belong. Assume that $x$'s defection from $Z'$ is critical. Prove that $y$'s defection from $Z'$ is also critical. (Hint: Assume, for contradiction, that $y$'s defection from $Z'$ is not critical. Consider the coalition $Z$ arrived at by deleting $x$ and $y$ from $Z'$.)
Exercises

26. Suppose that \( x \) and \( y \) are voters in a yes–no voting system and that \( x \approx y \). Suppose that \( Z' \) is a coalition that contains \( x \) but not \( y \). Let \( Z'' \) be the coalition resulting from replacing \( x \) by \( y \) in \( Z' \).
   (a) Prove that if \( Z' \) is winning, then \( Z'' \) is also winning.
   (b) Prove that if \( Z' \) is losing, then \( Z'' \) is also losing.
   (Hint for both parts: Let \( Z \) be the result of deleting \( x \) from \( Z' \) and then argue by contradiction.)

27. Assume that \( x \) and \( y \) are voters in a weighted yes–no voting system.
   (a) Assume that for some choice of weights and quota realizing the system, \( x \) and \( y \) have exactly the same weight. Prove that \( x \approx y \).
   (b) Assume that there are two choices of weights and quota realizing the yes–no voting system under consideration, one of which gives \( x \) more weight than \( y \) and one of which gives \( y \) more weight than \( x \). Prove that \( x \approx y \).
   [The converse of (a) and (b) is proved in Chapter 9, assuming the system is, in fact, weighted.]

28. In the U.S. federal system, let \( x \) be a member of the House and let \( y \) be a member of the Senate. Prove that \( x \) and \( y \) are incomparable.

29. In the U.S. federal system, let \( x \) be the vice president and let \( y \) be a member of the Senate. Prove that \( x \) and \( y \) are not incomparable.
   (One approach is to argue by contradiction.)

30. Using the results from Chapter 3, give an example of a yes–no voting system that is not weighted, but for which there are no incomparable voters.

31. Suppose \( x \) and \( y \) are voters in a yes–no voting system and suppose that their desirability is neither equal nor incomparable. Construct definitions (similar to what we did for incomparability) that formalize the notions that "\( x \) is more desirable than \( y \)" and "\( y \) is more desirable than \( x \)."