1.1 MODELING VIA DIFFERENTIAL EQUATIONS

The hardest part of using mathematics to study an application is the translation from real life into mathematical formalism. This translation is usually difficult because it involves the conversion of imprecise assumptions into very precise formulas. There is no way to avoid it. Modeling is difficult, and the best way to get good at it is the same way you get to play Carnegie Hall—practice, practice, practice.

What Is a Model?

It is important to remember that mathematical models are like other types of models. The goal is not to produce an exact copy of the "real" object but rather to give a representation of some aspect of the real thing. For example, a portrait of a person, a store mannequin, and a pig can all be models of a human being. None is a perfect copy of a human, but each has certain aspects in common with a human. The painting gives a description of what a particular person looks like; the mannequin wears clothes as a person does; and the pig is alive. Which of the three models is "best" depends on how we use the model—to remember old friends, to buy clothes, or to study biology.

We study mathematical models of systems that evolve over time, but they often depend on other variables as well. In fact, real-world systems can be notoriously complicated—the population of rabbits in Wyoming depends on the number of coyotes, the number of bobcats, the number of mountain lions, the number of mice (alternative food for the predators), farming practices, the weather, any number of rabbit diseases, etc. We can make a model of the rabbit population simple enough to understand only by making simplifying assumptions and lumping together effects that may or may not belong together.

Once we've built the model, we should compare predictions of the model with data from the system. If the model and the system agree, then we gain confidence that the assumptions we made in creating the model are reasonable, and we can use the model to make predictions. If the system and the model disagree, then we must study and improve our assumptions. In either case we learn more about the system by comparing it to the model.

The types of predictions that are reasonable depend on our assumptions. If our model is based on precise rules such as Newton's laws of motion or the rules of compound interest, then we can use the model to make very accurate quantitative predictions. If the assumptions are less precise or if the model is a simplified version of the system, then precise quantitative predictions would be silly. In this case we would use the model to make qualitative predictions such as "the population of rabbits in Wyoming will increase ...." The dividing line between qualitative and quantitative prediction is itself imprecise, but we will see that it is frequently better and easier to make qualitative use of even the most precise models.

Some hints for model building

The basic steps in creating the model are

Step 1 Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2 Completely describe the variables and parameters to be used in the model—"you can't tell the players without a program."

Step 3 Use the assumptions formulated in Step 1 to derive equations relating the quantities in Step 2.

Step 1 is the "science" step. In Step 1, we describe how we think the physical system works or, at least, what the most important aspects of the system are. In some cases these assumptions are fairly speculative, as, for example, "rabbits don't mind being overcrowded." In other cases the assumptions are quite precise and well accepted, such as "force is equal to the product of mass and acceleration." The quality of the assumptions determines the validity of the model and the situations to which the model is relevant. For example, some population models apply only to small populations in large environments, whereas others consider limited space and resources. Most important, we must avoid "hidden assumptions" that make the model seem mysterious or magical.

Step 2 is where we name the quantities to be studied and, if necessary, describe the units and scales involved. Leaving this step out is like deciding you will speak your own language without telling anyone what the words mean.

The quantities in our models fall into three basic categories: the **independent variable**, the **dependent variables**, and the **parameters**. In this book the independent variable is (almost) always time. Time is "independent" of any other quantity in the model. On the other hand, the dependent variables are quantities that are functions of the independent variable. For example, if we say that "position is a function of time," we mean that position is a variable that depends on time. We can vaguely state the goal of a model expressed in terms of a differential equation as "Describe the behavior of the dependent variable as the independent variable changes." For example, we may ask whether the dependent variable increases or decreases, or whether it oscillates or tends to a limit.

Parameters are quantities that don't change with time (or with the independent variable) but that can be adjusted (by natural causes or by a scientist running the experiment). For example, if we are studying the motion of a rocket, the initial mass of the rocket is a parameter. If we are studying the amount of ozone in the upper atmosphere, then the rate of release of fluorocarbons from refrigerators is a parameter. Determining how the behavior of the dependent variables changes when we adjust the parameters can be the most important aspect of the study of a model.

In Step 3 we create the equations. Most of the models we consider are expressed as differential equations. In other words, we expect to find derivatives in our equations. Look for phrases such as "rate of change of ..." or "rate of increase of ...," since rate of change is synonymous with derivative. Of course, also watch for "velocity" (derivative of position) and "acceleration" (derivative of velocity) in models from physics. The word *is* means "equals" and indicates where the equality lies. The phrase "A is proportional to B" means $A = kB$, where k is a proportionality constant (often a parameter in the model).

An important rule of thumb we use when formulating models is: *Always make the algebra as simple as possible.* For example, when modeling the velocity v of a cat falling from a tall building, we could assume:

- Air resistance increases as the cat's velocity increases.

This assumption says that air resistance provides a force that counteracts the force of gravity and that this force increases as the velocity v of the cat increases. We could choose kv or kv^2 for the air resistance term, where k is the friction coefficient, a parameter. Both expressions increase as v increases, so they satisfy the assumption. However, we most likely would try kv first because it is the simplest expression that satisfies the assumption. In fact, it turns out that kv yields a good model for falling bodies with low densities like snowflakes, but kv^2 is a more appropriate model for dense objects like raindrops.

Now we turn to a series of models of population growth based on various assumptions about the species involved. Our goal here is to study how to go from a set of assumptions to a model. These examples are not "state-of-the-art" models from population ecology, but they are good ones to consider initially. We also begin to describe the analytic, qualitative, and numerical techniques that we use to make predictions based on these models. Our approach is meant to be illustrative only; we discuss these mathematical techniques in much more detail throughout the entire book.

Unlimited Population Growth

An elementary model of population growth is based on the assumption that

- The rate of growth of the population is proportional to the size of the population.

Note that the rate of change of a population depends on only the size of the population and nothing else. In particular, limitations of space or resources have no effect. This assumption is reasonable for small populations in large environments—for example, the first few spots of mold on a piece of bread or the first European settlers in the United States.

Because the assumption is so simple, we expect the model to be simple as well. The quantities involved are

t = time (independent variable),

P = population (dependent variable), and

k = proportionality constant (parameter) between the rate of growth of the population and the size of the population.

The parameter k is often called the "growth-rate coefficient."

The units for these quantities depend on the application. If we are modeling the growth of mold on bread, then t might be measured in days and $P(t)$ might be either the area of bread covered by the mold or the weight of the mold. If we are talking about the European population of the United States, then t probably should be measured in years and $P(t)$ in millions of people. In this case we could let $t = 0$ correspond to any time we wanted. The year 1790 (the year of the first census) is a convenient choice.

Now let's express our assumption using this notation. The rate of growth of the population P is the derivative dP/dt. Being proportional to the population is expressed as the product, kP, of the population P and the proportionality constant k. Hence our assumption is expressed as the differential equation

$$\frac{dP}{dt} = kP.$$

In other words, the rate of change of P is proportional to P.

This equation is our first example of a differential equation. Associated with it are a number of adjectives that describe the type of differential equation that we are considering. In particular, it is a **first-order** equation because it contains only first derivatives of the dependent variable, and it is an **ordinary differential equation** because it does not contain partial derivatives. In this book we deal only with ordinary differential equations.

We have written this differential equation using the dP/dt Leibniz notation—the notation that we tend to use. However, there are many other ways to express the same differential equation. In particular, we could also write this equation as $P' = kP$ or as $\dot{P} = kP$. The "dot" notation is often used when the independent variable is time t.

What does the model predict?

More important than the adjectives or how the equation is written is what the equation tells us about the situation being modeled. Since $dP/dt = kP$ for some constant k, $dP/dt = 0$ if $P = 0$. Thus the constant function $P(t) = 0$ is a solution of the differential equation. This special type of solution is called an **equilibrium solution** because it is constant forever. In terms of the population model, it corresponds to a species that is nonexistent.

If $P(t_0) \neq 0$ at some time t_0, then at time $t = t_0$

$$\frac{dP}{dt} = k\,P(t_0) \neq 0.$$

As a consequence, the population is not constant. If $k > 0$ and $P(t_0) > 0$, we have

$$\frac{dP}{dt} = kP(t_0) > 0,$$

at time $t = t_0$ and the population is increasing (as one would expect). As t increases, $P(t)$ becomes larger, so dP/dt becomes larger. In turn, $P(t)$ increases even faster. That is, the rate of growth increases as the population increases. We therefore expect that the graph of the function $P(t)$ might look like Figure 1.1.

The value of $P(t)$ at $t = 0$ is called an **initial condition**. If we start with a different initial condition we get a different function $P(t)$ as is indicated in Figure 1.2. If $P(0)$ is negative (remembering $k > 0$), we then have $dP/dt < 0$ for $t = 0$, so $P(t)$ is initially decreasing. As t increases, $P(t)$ becomes more negative. The picture below the t-axis is the flip of the picture above, although this isn't "physically meaningful" because a negative population doesn't make much sense.

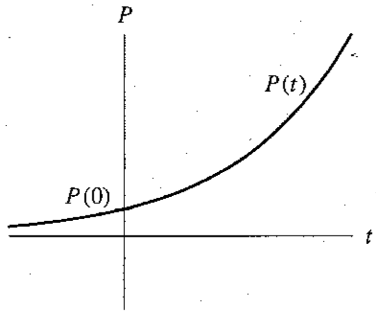

Figure 1.1
The graph of a function that satisfies the differential equation

$$\frac{dP}{dt} = kP.$$

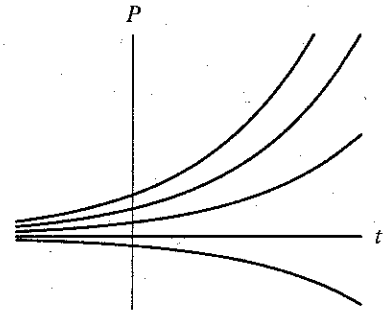

Figure 1.2
The graphs of several different functions that satisfy the differential equation $dP/dt = kP$. Each has a different value at $t = 0$.

Our analysis of the way in which $P(t)$ increases as t increases is called a **qualitative analysis** of the differential equation. If all we care about is whether the model predicts "population explosions," then we can answer "yes, as long as $P(0) > 0$."

Analytic solutions of the differential equation

If, on the other hand, we know the exact value P_0 of $P(0)$ and we want to predict the value of $P(10)$ or $P(100)$, then we need more precise information about the function $P(t)$. The pair of equations

$$\frac{dP}{dt} = kP, \quad P(0) = P_0,$$

is called an **initial-value problem**. A **solution** to the initial-value problem is a function $P(t)$ that satisfies both equations. That is,

$$\frac{dP}{dt} = kP \text{ for all } t \quad \text{and} \quad P(0) = P_0.$$

Consequently, to find a solution to this differential equation we must find a function $P(t)$ whose derivative is the product of k with $P(t)$. One (not very subtle) way to find such a function is to guess. In this case, it is relatively easy to guess the right form for $P(t)$ because we know that the derivative of an exponential function is essentially itself. (We can eliminate this guesswork by using the method of separation of variables, which we describe in the next section. But for now, let's just try the exponential and see where that leads us.) After a couple of tries with various forms of the exponential, we see that

$$P(t) = e^{kt}$$

is a function whose derivative, $dP/dt = ke^{kt}$, is the product of k with $P(t)$. But there are other possible solutions, since $P(t) = ce^{kt}$ (where c is a constant) yields $dP/dt = c(ke^{kt}) = k(ce^{kt}) = kP(t)$. Thus $dP/dt = kP$ for all t for any value of the constant c.

We have infinitely many solutions to the differential equation, one for each value of c. To determine which of these solutions is the correct one for the situation at hand, we use the given initial condition. We have

$$P_0 = P(0) = c \cdot e^{k \cdot 0} = c \cdot e^0 = c \cdot 1 = c.$$

Consequently, we should choose $c = P_0$, so a solution to the initial-value problem is

$$P(t) = P_0 e^{kt}.$$

We have obtained an actual formula for our solution, not just a qualitative picture of its graph.

The function $P(t)$ is called the solution to the initial-value problem as well as a **particular solution** of the differential equation. The collection of functions $P(t) = ce^{kt}$ is called the **general solution** of the differential equation because we can use it to find the particular solution corresponding to any initial-value problem. Figure 1.2 consists of the graphs of exponential functions of the form $P(t) = ce^{kt}$ with various values of the constant c, that is, with different initial values. In other words, it is a picture of the general solution to the differential equation.

The U.S. Population

As an example of how this model can be used, consider the U.S. census figures since 1790 given in Table 1.1.

Table 1.1
U.S. census figures, in millions of people (see www.census.gov)

Year	t	Actual	$P(t) = 3.9e^{0.03067t}$	Year	t	Actual	$P(t) = 3.9e^{0.03067t}$
1790	0	3.9	3.9	1930	140	122	286
1800	10	5.3	5.3	1940	150	131	388
1810	20	7.2	7.2	1950	160	151	528
1820	30	9.6	9.8	1960	170	179	717
1830	40	12	13	1970	180	203	975
1840	50	17	18	1980	190	226	1,320
1850	60	23	25	1990	200	249	1,800
1860	70	31	33	2000	210	281	2,450
1870	80	38	45	2010	220		3,320
1880	90	50	62	2020	230		4,520
1890	100	62	84	2030	240		6,140
1900	110	75	114	2040	250		8,340
1910	120	91	155	2050	260		11,300
1920	130	105	210				

Let's see how well the unlimited growth model fits this data. We measure time in years and the population $P(t)$ in millions of people. We also let $t = 0$ be the year 1790, so the initial condition is $P(0) = 3.9$. The corresponding initial-value problem

$$\frac{dP}{dt} = kP, \quad P(0) = 3.9,$$

has $P(t) = 3.9e^{kt}$ as a solution. We cannot use this model to make predictions yet because we don't know the value of k. However, we are assuming that k is a constant, so we can use the initial condition along with the population in the year 1800 to estimate k. If we set

$$5.3 = P(10) = 3.9e^{k\cdot 10},$$

then we have

$$e^{k\cdot 10} = \frac{5.3}{3.9}$$

$$10k = \ln\left(\frac{5.3}{3.9}\right)$$

$$k \approx 0.03067.$$

Thus our model predicts that the United States population is given by

$$P(t) = 3.9e^{0.03067t}.$$

As we see from Figure 1.3, this model of $P(t)$ does a decent job of predicting the population until roughly 1860, but after 1860 the prediction is much too large. (Table 1.1 includes a comparison of the predicted values to the actual data.)

Our model is fairly good provided the population is relatively small. However, as time goes on, the model predicts that the population will continue to grow without any limits, and obviously, this cannot happen in the real world. Consequently, if we want a model that is accurate over a large time scale, we should account for the fact that populations exist in a finite amount of space and with limited resources.

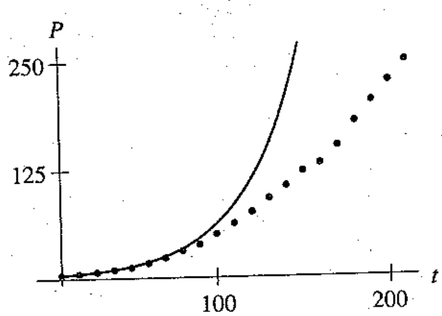

Figure 1.3
The dots represent actual census data and the solid line is the solution of the exponential growth model

$$\frac{dP}{dt} = 0.03067P.$$

Time t is measured in years since the year 1790.

10. Model radioactive decay using the notation

$$t = \text{time (independent variable)},$$
$$r(t) = \text{amount of particular radioactive isotope present at time } t \text{ (dependent variable)},$$
$$-\lambda = \text{decay rate (parameter)}.$$

Note that the minus sign is used so that $\lambda > 0$.

(a) Using this notation, write a model for the decay of a particular radioactive isotope.

(b) If the amount of the isotope present at $t = 0$ is r_0, state the corresponding initial-value problem for the model in part (a).

11. The **half-life** of a radioactive isotope is the amount of time it takes for a quantity of radioactive material to decay to one-half of its original amount.

(a) The half-life of Carbon 14 (C-14) is 5230 years. Determine the decay-rate parameter λ for C-14.

(b) The half-life of Iodine 131 (I-131) is 8 days. Determine the decay-rate parameter for I-131.

(c) What are the units of the decay-rate parameters in parts (a) and (b)?

(d) To determine the half-life of an isotope, we could start with 1000 atoms of the isotope and measure the amount of time it takes 500 of them to decay, or we could start with 10,000 atoms of the isotope and measure the amount of time it takes 5000 of them to decay. Will we get the same answer? Why?

12. Carbon dating is a method of determining the time elapsed since the death of organic material. The assumptions implicit in carbon dating are that

- Carbon 14 (C-14) makes up a constant proportion of the carbon that living matter ingests on a regular basis, and
- once the matter dies, the C-14 present decays, but no new carbon is added to the matter.

Hence, by measuring the amount of C-14 still in the organic matter and comparing it to the amount of C-14 typically found in living matter, a "time since death" can be approximated. Using the decay-rate parameter you computed in Exercise 11, determine the time since death if

(a) 88% of the original C-14 is still in the material.

(b) 12% of the original C-14 is still in the material.

(c) 2% of the original C-14 is still in the material.

(d) 98% of the original C-14 is still in the material.

Remark: There has been speculation that the amount of C-14 available to living creatures has not been exactly constant over long periods (thousands of years). This makes accurate dates much trickier to determine.