Math 35
Second Midterm

Show your work. Correct answers with no justification may receive little or no credit. No calculators are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space.

Problem 1. In this problem, let $D_R$ denote the disk of radius $R$ centered at the origin in $\mathbb{R}^2$. That is to say, $D_R = \{(x, y) \mid x^2 + y^2 \leq R^2\}$. Similarly, let $S_R = [-R, R] \times [-R, R]$ be the square centered at the origin with sides which are parallel to the coordinate axes and of length $2R$.

a) (10 points) Calculate $\iint_{D_R} e^{-(x^2+y^2)} \, dA$.

Solution.

$$\iint_{D_R} e^{-(x^2+y^2)} \, dA = \int_0^{2\pi} \int_0^R e^{-r^2} r \, dr \, d\theta = -2\pi \left. \frac{1}{2} e^{-r^2} \right|_{r=0}^R = \pi \left(1 - e^{-R^2}\right).$$

b) (5 points) Express $\iint_{S_R} e^{-(x^2+y^2)} \, dA$ in terms of the quantity $\int_{-R}^R e^{-x^2} \, dx$.

Solution.

$$\iint_{S_R} e^{-(x^2+y^2)} \, dA = \int_{-R}^R \int_{-R}^R e^{-x^2} e^{-y^2} \, dx \, dy = \left(\int_{-R}^R e^{-x^2} \, dx\right) \left(\int_{-R}^R e^{-y^2} \, dy\right) = \left(\int_{-R}^R e^{-x^2} \, dx\right)^2.$$

d) (5 points) Justify the following inequalities:

$$\iint_{D_R} e^{-(x^2+y^2)} \, dA < \iint_{S_R} e^{-(x^2+y^2)} \, dA < \iint_{D_{2R}} e^{-(x^2+y^2)} \, dA.$$
Solution. \( e^{-(x^2+y^2)} \) is positive for all \((x, y)\) and \( D_R \subset S_R \subset D_{\sqrt{2}R} \). (\( S_R \) is the circle circumscribed around \( D_R \) while \( D_{\sqrt{2}R} \) is a square circumscribed around \( S_R \).) \( \square \)

d) (5 points) Use the previous parts of this problem to deduce the value of \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \). (Here, you may assume without proof that all the following limits exist and are finite: \( \lim_{R \to \infty} \int_{-R}^{R} e^{-x^2} \, dx \), \( \lim_{R \to \infty} \iint_{S_R} e^{-(x^2+y^2)} \, dA \), and \( \lim_{R \to \infty} \iint_{D_R} e^{-(x^2+y^2)} \, dA \).)

Solution. From part c we see that

\[
\lim_{R \to \infty} \iint_{D_R} e^{-(x^2+y^2)} \, dA = \lim_{R \to \infty} \iint_{S_R} e^{-(x^2+y^2)} \, dA.
\]

From part b we see that

\[
\lim_{R \to \infty} \iint_{S_R} e^{-(x^2+y^2)} \, dA = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2.
\]

From part a we see that

\[
\lim_{R \to \infty} \iint_{D_R} e^{-(x^2+y^2)} \, dA = \pi.
\]

Therefore \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \). \( \square \)

Problem 2. In this problem, let \( f(x, y) = 6x - 8y - x^2 - y^2 \).

a) (5 points) Find the gradient and the Hessian of \( f \).

Solution.

\[
\nabla f = (6 - 2x, -8 - 2y)
\]

\[
Hf = \begin{bmatrix}
-2 & 0 \\
0 & -2
\end{bmatrix}.
\]

\( \square \)

b) (5 points) Find and classify the critical points of \( f \).

Solution. Using the formulas from part a, we see that the only critical point is located at \((3, -4)\) and that the second derivative test shows it to be a local maximum. \( \square \)
c) (10 points) Use the method of Lagrange Multipliers to find the maximum and minimum values that $f$ takes on the circle $x^2 + y^2 = 100$.

Solution. Letting $g = x^2 + y^2$, the conditions $\nabla f = \lambda \nabla g$ and $g = 100$ become the system of equations

$$
\begin{align*}
6 - 2x &= 2\lambda x \\
-8 - 2y &= 2\lambda y \\
100 &= x^2 + y^2
\end{align*}
$$

(1) \hspace{5cm} (2) \hspace{5cm} (3)

Solving equation (1) for $x$ and equation (2) for $y$, we see that $y = -\frac{4}{x}$. Substituting this into equation (3) and solving for $x^2$ gives $x^2 = 36$, so $x = \pm 6$. Thus the two constrained critical points we get are $(6, -8)$ and $(-6, 8)$. Since $f(6, -8) = 0$ and $f(-6, 8) = -200$, the first is a maximum and the second is a minimum.

\[\square\]

d) (5 points) Use the previous parts of this problem to find the maximum and minimum values that $f$ takes on the disk $D_{10} = \{(x, y) | x^2 + y^2 \leq 100\}$.

Solution. Whatever extreme values $f$ takes on $D_{10}$, they must either occur in the interior, and thus at the critical point $(3, -4)$, found in part b, or occur on the boundary and thus at one of the constrained critical points, $(6, -8)$ or $(-8, 8)$. The values are these locations are $f(3, -4) = 25$, $f(6, -8) = 0$, and $f(-6, 8) = -200$. Thus the first of these values is the maximum and the last is the minimum.

\[\square\]

Problem 3. In this problem, let $\mathcal{H} = \{(x, y, z) | x^2 + y^2 + z^2 \leq 25 \text{ and } x \geq 0\}$, be the hemisphere of radius 5 centered at the origin and having positive $x$ coordinate. Let $f : \mathcal{H} \rightarrow \mathbb{R}$ be an unknown scalar function.

a) (9 points) Express $\iiint_{\mathcal{H}} f \, dV$ as an iterated integral in rectangular $(x, y, z)$ coordinates.

Solution. Several orders of integration are possible. Here is one:

$$
\int_{z=-5}^{z=5} \int_{y=-\sqrt{25-z^2}}^{y=\sqrt{25-z^2}} \int_{x=0}^{x=\sqrt{25-y^2-z^2}} f(x, y, z) \, dx \, dy \, dz.
$$

\[\square\]

b) (8 points) Express $\iiint_{\mathcal{H}} f \, dV$ as an iterated integral in cylindrical $(r, \theta, z)$ coordinates.
Solution. Note that on the spherical part of the boundary of $H$ we have $r^2 + z^2 = 25$. Again more than one order is possible. Here is one:

$$
\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=5} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta
$$

\[ \square \]

c) (8 points) Express $\iiint_H f \, dV$ as an iterated integral in spherical $(\rho, \phi, \theta)$ coordinates.

Solution.

$$
\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=5} f(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
$$

\[ \square \]

Problem 4. Let $C$ be a simple closed curve going counter-clockwise around a region $D$ in the plane.

a) (10 points) Express the integral $\oint_C M \, dx$ as a double integral over $D$.

Solution. Green's theorem with $N = 0$ gives:

$$
\oint_C M \, dx = - \iint_D \frac{\partial M}{\partial y} \, dy \, dx
$$

\[ \square \]

b) (15 points) Find $M$ so that $\oint_C M \, dx$ gives the $x$-coordinate of the centroid of $D$. (You may assume that the area, $A$, of $D$ is known.)

Solution. What we need is to find $M$ so that

$$
- \iint_D \frac{\partial M}{\partial y} \, dy \, dx = \frac{1}{A} \iint_D x \, dy \, dx.
$$

This will be accomplished if $\frac{\partial M}{\partial y} = -\frac{x}{A}$. Thus $M = -xy/A$ works just fine as will any function of the form $-\frac{xy}{A} + g(x)$.