This is a closed book, in-class exam. Calculators are not allowed. You may use a pre-prepared half-sheet (one-sided) of notes.

Write all of your work in the space provided. If you run out of space, use the backs of the pages. If you use the backs of the pages, label your work with the appropriate problem number.

Show your work. You should always give at least a brief justification for each answer; correct answers with no justification may recieve little or no credit. No uncalled-for simplification is required.

Write your name below, and also on any of the pages that become detached from this cover-page as you work the exam.

Name: ________________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
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<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>100</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. (20 points) Let $D$ be the region in the first quadrant of the $xy$-plane bounded by the ellipse $x^2 + 4y^2 = 4$. Thus, $D$ is the quarter ellipse drawn below.

Find $\int\int_D xy \, dx \, dy$ using the substitution $x = \sqrt{u}$, $y = \sqrt{v}$.

Solution: With the specified substitution, the corresponding region $D^*$ in $uv$ space is specified by $u, v \geq 0$ and $u + 4v \leq 4$.

We have $\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{pmatrix} \frac{1}{2}u^{-1/2} & 0 \\ 0 & \frac{1}{2}v^{-1/2} \end{pmatrix} = \frac{1}{4}(uv)^{-1/2}$. Thus, the change of variables formula gives us

$$\int\int_D xy \, dx \, dy = \int\int_{D^*} \sqrt{u} \sqrt{v} \frac{1}{4}(uv)^{-1/2} = \frac{1}{4} \int\int_{D^*} du \, dv = \frac{1}{4} \cdot \frac{1}{2} = 1.$$

Here we have used the well-known formula for the area of a triangle—we could equally well computed $\int_{u=0}^{4} \int_{v=0}^{1 - \frac{x}{4}} dv \, du = 2$.

2. (a) (10 points) Use Lagrange multipliers to find the maximum value taken by $f(x, y) = y^2 - x^3 + x$ on the circle $x^2 + y^2 = 1$. (As often happens, this problem can be done without Lagrange multipliers, but you won’t get credit if you do it that way.)
Solution: We need to solve the three equations in three unknowns given by
\[ \nabla f = \lambda \nabla (x^2 + y^2) \] and \( x^2 + y^2 = 1 \). We first write them all out together.

\[-3x^2 + 1 = \lambda 2x \]
\[2y = \lambda 2y \]
\[x^2 + y^2 = 1 \]

If \( y = 0 \), the second of these equations is automatically satisfied, the third equation forces \( x = \pm 1 \) and the first one then determines \( \lambda \). On the other hand, if \( y \) is not zero, then the second equation implies that \( \lambda = 1 \), and so the first equation becomes a simple quadratic equation with roots \( x = -1 \) or \( x = 1/3 \). The third equation then determines \( y \). All in all we have four constrained critical points \((\pm 1, 0)\) and \((\frac{1}{3}, \pm \frac{2\sqrt{2}}{3})\). We calculate \( f(\pm 1, 0) = 0 \) and \( f\left(\frac{1}{3}, \pm \frac{2\sqrt{2}}{3}\right) = \frac{32}{27} \). Thus, the maximum value \( f \) takes on the circle is \( \frac{32}{27} \).

(b) (10 points) Find and classify the critical points \( f \) has in the interior of the circle.

Solution: Here we solve \( \nabla f = 0 \). That is to say, we need \( 2y = 0 \) and \( 3x^2 = 1 \). Thus there are two critical points \((\pm \sqrt{\frac{1}{3}}, 0)\). To classify these, we look at the Hessian. We compute \( Hf = \begin{bmatrix} -6x & 0 \\ 0 & 2 \end{bmatrix} \). At \( x = \sqrt{\frac{1}{3}} \) we see that \( Hf \) is neither positive nor negative definite, while at \( x = -\sqrt{\frac{1}{3}} \) we see that \( Hf \) is positive definite. Thus \( \left(\sqrt{\frac{1}{3}}, 0\right) \) is the location of a saddle and \( \left(-\sqrt{\frac{1}{3}}, 0\right) \) is the location of a relative minimum. We are entirely done with this problem, but I will add a nice picture below which will be useful if I ever want to use this problem as an example in class. The dotted lines are evenly spaced contours of \( f \). The shaded circle is \( x^2 + y^2 = 1 \). The marked points are the constrained and unconstrained critical points. The solid lines are additional.
contours through those critical points.

3. A wire of uniform density lies in the $xy$-plane along the graph of the function $y = \frac{2}{3}(x - 1)^{3/2}$ in the range $1 \leq x \leq 4$.

(a) (10 points) Find the length of the wire.

**Solution:** We parameterize the wire by $(x, y) = (t, \frac{2}{3}(t - 1)^{3/2})$. We then have $\vec{x} = (1, \sqrt{t - 1})$ and $\|\vec{x}\| = \sqrt{t}$. Thus, the length is given by the following calculation.

$$\int_{1}^{4} ds = \int_{1}^{4} \sqrt{t} \, dt = \left. \frac{2}{3} t^{3/2} \right|_{1}^{4} = \frac{14}{3}. $$

(b) (10 points) Find the $x$-coordinate of the center of mass of the wire.

**Solution:** Taking up where we left off we compute as follows.

$$x_{\text{com}} = \frac{\int_{1}^{4} x \, ds}{\text{length}} = \frac{3}{14} \int_{1}^{4} t^{1/2} \, dt = \frac{93}{35}. $$

4. Of the following two vector fields

$$\vec{F} = (y + z, x + 2y + z, y)$$

$$\vec{G} = (y, x + 2y + z, y)$$
one is conservative and the other is not.

(a) (10 points) Determine which is which.

**Solution:** We calculate:

\[
\vec{\nabla} \times \vec{F} = \text{det} \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
D_x & D_y & D_z \\
y + z & x + 2y + z & y
\end{pmatrix} = \vec{j} \neq \vec{0}.
\]

Thus, \(\vec{F}\) is not conservative. If we trust the examiner, we just say that \(\vec{G}\) is conservative at this point, but it is wise to check anyway:

\[
\vec{\nabla} \times \vec{G} = \text{det} \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
D_x & D_y & D_z \\
y & x + 2y + z & y
\end{pmatrix} = \vec{0}.
\]

Thus \(\vec{G}\) is conservative.

(b) (10 points) For the conservative one, find a scalar potential.

**Solution:** We seek a \(g\) with \(\vec{\nabla} g = \vec{G}\). We start by considering the derivatives with respect to \(x\). To obtain \(\frac{\partial g}{\partial x} = y\), we must have \(g = xy + g_1(y, z)\), where \(g_1\) remains to be determined. Considering derivatives with respect to \(y\), we see that to obtain \(\frac{\partial g}{\partial y} = x + 2y + z\), we must have \(x + \frac{\partial g_1}{\partial y} = x + 2y + z\). We deduce that \(g_1 = y^2 + zy + g_2(z)\), where \(g_2\) remains to be determined. Finally, in order to obtain \(\frac{\partial g}{\partial z} = y\) we must have \(y + g_2' = y\) so \(g_2\) should be constant. Thus \(g = xy + y^2 + zy + c\) will be a scalar potential for \(\vec{G}\).

5. (20 points) Use Green’s theorem to calculate the area within the circle \(x^2 + y^2 = 4\) bounded on the left by the line \(x = 1\). It is possible to do this problem without using Green’s theorem, but that won’t get you any credit. In your solution, clearly show
how you use Green’s theorem.

Solution: The points of intersection of the line and the circle are at \((1, \pm \sqrt{3})\). The angles in polar coordinates of these points are \(\pm \frac{\pi}{3}\). Green’s theorem states

\[
\oint_{\partial D} M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.
\]

In order to have the right hand side come out to area, we use the trick we saw in class and let \(N = \frac{1}{2} x\) and \(M = -\frac{1}{2} y\). Thus Green’s theorem tells us that the area is given by 

\[
\frac{1}{2} \oint_{D} (x \, dy - y \, dx) = \frac{1}{2} \iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.
\]

On the circle we take \(x = 2 \cos \theta\) and \(y = 2 \sin \theta\), so \(dx = -2 \sin \theta \, d\theta\) and \(dy = 2 \cos \theta \, d\theta\). We then calculate

\[
\frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (x \, dy - y \, dx) = \frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} 4 \, d\theta = \frac{4}{3} \pi.
\]

Meanwhile, on the segment (oriented from top to bottom) we take \(x = 1\) and use \(y\) as our parameter (and \(dx = 0\)) to get

\[
\frac{1}{2} \int_{\text{segment}} (x \, dy - y \, dx) = \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} dy = -\sqrt{3}.
\]

Putting the two pieces together we see the area is \(\frac{4}{3} \pi - \sqrt{3}\).