Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Show your work. Correct answers with no justification may receive little or no credit. This is a closed book exam, but you may use a pre-prepared 8.5 by 11 inch (one-sided) sheet of notes. No calculators are allowed. No uncalled-for simplification is required. Write your name below, and also on any of the pages that become detached from this cover-page as you work the exam.

Name: ________________________________

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1. Suppose a particle in $\mathbb{R}^2$ has position $\vec{x}(t) = \left( \frac{t^3}{3} - t, t^2 \right)$ at time $t$.

(a) (5 points) Find the velocity of this particle as a function of $t$.

**Solution:** $\vec{v}(t) = \dot{\vec{x}}(t) = (t^2 - 1, 2t)$.

(b) (5 points) Find the speed of this particle as a function of $t$.

**Solution:** speed $= \dot{s}(t) = \|\dot{\vec{x}}(t)\| = \sqrt{(t^2 - 1)^2 + 4t^2} = \sqrt{(t^2 + 1)^2} = t^2 + 1$.

(c) (5 points) Find the acceleration of this particle as a function of $t$.

**Solution:** $\vec{a}(t) = \ddot{\vec{x}}(t) = (2t, 2)$.

(d) (5 points) Find the total distance traveled (arc-length) as a function of $t$.

**Solution:** Assuming that we start measuring the distance traveled at time $t = 0$, we get the arc-length function

$$\int_0^t \dot{s}(\tau) \, d\tau = \int_0^t (\tau^2 + 1) \, d\tau = \frac{t^3}{3} + t.$$

2. Let $\vec{F}(t) = (e^{yz}, \cos(xz), xy)$.

(a) (10 points) Find the divergence of $\vec{F}$.

**Solution:**

$$\text{dif } \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial e^{yz}}{\partial x} + \frac{\partial \cos(xz)}{\partial y} + \frac{\partial xy}{\partial z} = 0 + 0 + 0 = 0.$$

(b) (10 points) Find the curl of $\vec{F}$.

**Solution:**

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & \cos(xz) & xy \end{bmatrix}$$

$$= (x + x \cos(xz), ye^{yz} - y, -z \cos(xz) - ze^{yz}).$$
3. (20 points) If \( f(x, y) = \sin(xy) \) and \( x = uv \) and \( y = u + v \), show how the chain rule in matrix form allows you to calculate \( \frac{\partial f}{\partial u} \) and \( \frac{\partial f}{\partial v} \). (Note: These derivatives are not too hard to calculate in other ways, but that is not the point of this problem; rather I want you to show that you understand how the chain rule in matrix form applies in this context.)

**Solution:** Define a function \( \bar{g} \) by \( \bar{g}(u, v) = (x, y) = (uv, u + v) \). The required partial derivatives are then the entries in the gradient of the composite function \( f \circ \bar{g} \). Thus we compute:

\[
\left( \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right) = D(f \circ \bar{g}) = Df|_{\bar{g}(u,v)} \cdot Dg = (y \cos(xy), x \cos(xy))|_{(x,y)=(uv,u+v)} \cdot \begin{bmatrix} v & u \\ 1 & 1 \end{bmatrix}
\]

\[
= ((u + v) \cos(\cos(xy)), u \cos(xy)) \cdot \begin{bmatrix} v & u \\ 1 & 1 \end{bmatrix}
\]

\[
= ((u^2 + 2uv) \cos(2uv + 2u^2), (u^2 + 2uv) \cos(2uv + 2u^2)).
\]

4. (20 points) Considering the ellipse \( x^2 + 2xy + 3y^2 = 6 \) to be a level curve of the function \( f(x, y) = x^2 + 2xy + 3y^2 \), and using a gradient, find the points on this ellipse where the tangent (to the ellipse) is parallel to the vector \((2, -1)\).

**Solution:** We have \( \nabla f = (2x + 2y, 2x + 6y) \). The gradient is perpendicular to the level curves, so we wish to find a point where the gradient is perpendicular to \((2, -1)\). This gives us the equation \((2x + 2y, 2x + 6y) \cdot (2, -1) = 0\). That is to say \(2x - 2y = 0\), so \(x = y\). Plugging this into the equation of the ellipse gives us \(6x^2 = 6\), so \(x = \pm 1\). Thus the two points satisfying the required conditions are \((1, 1)\), and \((-1, -1)\).

5. (20 points) Evaluate the limit

\[
\lim_{(x,y) \to (0,0)} \frac{x^3 - xy^2}{x^2 + y^2}
\]

or give a brief, clear argument showing that the limit does not exist.

**Solution:** Write \(x = r \cos \theta\) and \(y = r \sin \theta\). We then have

\[
\lim_{(x,y) \to (0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^3 \theta - r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \to 0} r (\cos^3 \theta - \cos \theta \sin^2 \theta) = 0.
\]

The last equality here follows because the parenthesized trigonometric term cannot possibly be any bigger than two in absolute value, since sine and cosine are never bigger than one in absolute value.