Math 34
First Midterm

Show your work. Correct answers with no justification may receive little or no credit. No calculators are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space.

One way of specifying spherical coordinates is by the following formulae.
\[
  x = \rho \sin \varphi \cos \theta \\
  y = \rho \sin \varphi \sin \theta \\
  z = \rho \cos \varphi
\]

\[
\rho = \sqrt{x^2 + y^2 + z^2} \\
\varphi = \arctan \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \\
\theta = \arctan \left( \frac{y}{x} \right)
\]

Use these formulae as necessary in the following problems 1 through 4. In these problems also let the function \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by
\[
\Phi(\rho, \varphi, \theta) := (x(\rho, \varphi, \theta), y(\rho, \varphi, \theta), z(\rho, \varphi, \theta)) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).
\]

Problem 1. Calculate the matrix derivative, \( D\Phi \).

\textit{Solution.}
\[
D\Phi = \begin{bmatrix}
\sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\
\sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\
\cos \varphi & -\rho \sin \varphi & 0
\end{bmatrix}
\]

\( \square \)

Problem 2. Calculate the divergence, \( \nabla \cdot \Phi \).

\textit{Solution.}
\[
\nabla \cdot \Phi = \frac{\partial x}{\partial \rho} + \frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \theta} = \text{trace } D\Phi = \sin \varphi \cos \theta + \rho \cos \varphi \sin \theta.
\]

\( \square \)
Problem 3. Calculate the gradient, $\nabla \rho$.

Solution.

$$\nabla \rho = \nabla \sqrt{x^2 + y^2 + z^2} = \left(\frac{2x}{2\sqrt{x^2 + y^2 + z^2}}, \frac{2y}{2\sqrt{x^2 + y^2 + z^2}}, \frac{2z}{2\sqrt{x^2 + y^2 + z^2}}\right).$$

Problem 4. Suppose that $f(x, y, z) = ye^x + z$. Use the chain rule to calculate $\frac{\partial f}{\partial \phi}$. Express your answer in terms of $\rho$, $\phi$, and $\theta$.

Solution.

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \phi}$$

$$= ye^x \rho \cos \phi \cos \theta + e^x \rho \cos \phi \sin \theta - \rho \sin \phi$$

$$= \rho \sin \phi \sin \theta e^{\phi \sin \theta} \rho \cos \phi \cos \theta + e^{\phi \sin \theta} \rho \cos \phi \sin \theta - \rho \sin \phi$$

In the following problems 5 and 6, let $\vec{x}$ be the path defined by

$$\vec{x}(t) = t\hat{i} + \frac{2}{3}(2t + 1)^{\frac{3}{2}}\hat{j}, \quad 0 \leq t \leq 4.$$

Problem 5. Calculate the velocity, acceleration, and unit tangent vector for this path.

Solution.

$$\text{velocity} = \dot{\vec{x}} = 2t\hat{i} + 2(2t + 1)^{\frac{3}{2}}\hat{j} = 2\left(t, \sqrt{2t + 1}\right)$$

$$\text{acceleration} = \ddot{\vec{x}} = 2\hat{i} + 2 \frac{1}{2}(2t + 1)^{-\frac{1}{2}} = 2(1, (2t + 1)^{-1/2})$$

$$\text{unit tangent} = \frac{\dot{\vec{x}}}{\|\dot{\vec{x}}\|} = \frac{2 \left(t, \sqrt{2t + 1}\right)}{2 \left(\sqrt{t^2 + 2t + 1}\right)} = \left(\frac{t}{t + 1}, \frac{\sqrt{2t + 1}}{t + 1}\right)$$

Problem 6. Calculate the length of this path.
Solution. In the previous problem, we calculated the speed as $\|\dot{x}\| = 2(t + 1)$. It follows that
\[
\text{length} = \int_0^1 2(t + 1) \, dt = 2 \left[ \frac{t^2}{2} + t \right]_0^1 = 24.
\]

In the following problems 7 and 8, calculate an equation for the plane tangent to the surface $x^2 + y^3 + z^4 = 3$ at $(1, 1, 1)$ two ways:

Problem 7. By solving for $z$ in terms of $x$ and $y$ and finding the linear approximation.

Solution. We have $z = (3 - x^2 - y^3)^{1/4}$. Thus we have
\[
\left[ \frac{\partial z}{\partial x} \right]_{1,1} = \left[ \frac{1}{4} (3 - x^2 - y^3)^{-3/4} (-2x) \right]_{1,1} = -\frac{1}{2},
\]
\[
\left[ \frac{\partial z}{\partial y} \right]_{1,1} = \left[ \frac{1}{4} (3 - x^2 - y^3)^{-3/4} (-3y^2) \right]_{1,1} = -\frac{3}{4}.
\]

Thus the linear approximation at $(x, y) = (1, 1)$ is given by
\[
z = 1 - \frac{1}{2}(x - 1) - \frac{3}{4}(y - 1).
\]

Problem 8. By recognizing the surface as the level surface of a certain function $\mathbb{R}^3 \to \mathbb{R}$ and using the relationship between gradients and level surfaces.

Solution. Put $F(x, y, z) = x^2 + y^3 + z^4$. Then
\[
\left[ \vec{\nabla} F \right]_{1,1,1} = \left[ (2x, 3y^2, 4z^3) \right]_{1,1,1} = (2, 3, 4).
\]

So the desired plane passes through $(1, 1, 1)$ and is perpendicular to $(2, 3, 4)$. Thus the equation is
\[
((x, y, z) - (1, 1, 1)) \cdot (2, 3, 4) = 0.
\]

That is to say $2(x - 1) + 3(y - 1) + 4(z - 1) = 0$. □

In the following two problems, 9 and 10, either evaluate the given limit or explain why it does not exist.
Problem 9.

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 + 2xy + y^2}{x+y} \]

**Solution.**

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 + 2xy + y^2}{x+y} = \lim_{(x,y) \to (0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y) \to (0,0)} (x+y) = 0. \]

\[ \square \]

Problem 10.

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \]

**Solution.** This limit does not exist, because the limit along the \( x \)-axis is not equal to the limit along the \( y \)-axis:

\[ \lim_{x \to 0 \atop y=0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = 1 \]

\[ \lim_{y \to 0 \atop x=0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y \to 0} \frac{-y^2}{y^2} = -1. \]

\[ \square \]