Math 34
Second Midterm

Show your work. Correct answers with no justification may receive little or no credit. No calculators are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space.

Each part of each problem is worth ten points.

Problem 1. In this problem, we consider the integral
\[ \int_{0}^{1} \int_{x^2}^{1} x \sqrt{1 - y^2} \, dy \, dx. \]

a) (10 points) Sketch the region of integration.

Solution.

Note that the integrand is non-negative in the specified domain, so we know that in the end it should evaluate to a positive number. □

b) (10 points) Rewrite the integral with the order of integration reversed.

Solution.

\[ \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} x \sqrt{1 - y^2} \, dx \, dy. \]

□

c) (10 points) One of the two orders of integration requires nothing worse than a simple substitution to evaluate fully. Decide which is which and do the integral.

Solution.

\[ \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} x \sqrt{1 - y^2} \, dx \, dy = \int_{y=0}^{1} \sqrt{1 - y^2} \left[ \frac{x^2}{2} \right]_{x=0}^{\sqrt{y}} \, dy = -\frac{1}{6} \left( 1 - y^2 \right)^{3/2} \bigg|_{0}^{1} = \frac{1}{6}. \]

□

Problem 2. In this problem, we consider the part of the sphere of radius $R$ which lies in the first octant; call this region $W$. You should feel free to use the fact the volume of the whole sphere is $\frac{4}{3} \pi R^3$.

a) (10 points) Write down a formula involving an iterated integral in cylindrical coordinates which gives the $z$ coordinate of the centroid of $W$.

Solution. In rectangular coordinates, we have, $x^2 + y^2 + z^2 = R^2$ on the boundary of the sphere. In cylindrical coordinates, this gives $z = \sqrt{R^2 - r^2}$. □
Furthermore, the volume of this eighth of the sphere is $\frac{1}{8} \frac{4}{3} \pi R^3$. Thus, the desired formula for the $z$ coordinate of the centroid is

$$\frac{1}{\pi R^3} \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \int_{z=0}^{\sqrt{R^2-r^2}} zr \, dz \, dr \, d\theta.$$ 

b) (10 points) Evaluate the integral you gave in part a of this problem.

Solution. (Note that before doing the integral, we can expect, based on our experience, that the answer is proportional to $R$, and that it should give a value greater than 0 and less than $R/2$.)

\[
\frac{6}{\pi R^3} \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \int_{z=0}^{\sqrt{R^2-r^2}} zr \, dz \, dr \, d\theta = \frac{\pi}{2} \frac{6}{\pi R^3} \int_{r=0}^{R} r \left( \frac{z^2}{2} \right)_{z=0}^{\sqrt{R^2-r^2}} \, dr
\]

\[
= \frac{3}{R^3} \int_{r=0}^{R} r (R^2 - r^2) \, dr = \frac{3}{R^3} \left[ \frac{R^2 - r^2}{2} \right]_{r=0}^{R} = \frac{3}{R^3} \left( \frac{R^4}{4} - \frac{R^2}{4} \right) = \frac{3}{8} R.
\]

□

c) (10 points) What are the $x$ and $y$ coordinates of the centroid of $W$? (You shouldn’t have to do any more integration.)

Solution. The figure remains fixed when we interchange the various positive axes. So, by symmetry the $x$ and $y$ coordinates of the center of mass should also be $\frac{3}{8} R$, too. □

Problem 3. In this problem we consider the curve, $C$, formed by the intersection of the cylinder $x^2 + z^2 = 9$ with the plane $x + y + z = 1$. Orient this curve so that it is clockwise when viewed from a point like $(0,10,0)$ which is on the positive $y$-axis some distance away from the curve.

a) (10 points) Find a parameterization of $C$.

Solution. Since $C$ is the intersection of a plane with a cylinder centered along the $y$-axis, its projection on the $xz$ plane is a circle, and we can parameterize $C$ by parameterizing that circle. Since $y = 1 - x - z$ on the plane, the result is $\vec{X} = (x, y, z) = (3 \cos \theta, 1 - 3 \cos \theta - 3 \sin \theta, 3 \sin \theta)$. Here $\theta \in [0, 2\pi]$. We have the desired orientation since in the projection on the $xz$ plane we start at the $x$-axis, when $\theta = 0$, and next pass through the $z$-axis, when $\theta = \pi/2$. (I did not expect you to need to draw an accurate picture in order to do this problem, but here is one anyway, to help you see what is going on.)

□

b) (10 points) Compute the integral

$$\oint_C x \, dx + y \, dy + z \, dz.$$
\[ \int_C (x \, dx + y \, dy + z \, dz) = \int_0^\pi \left[ (3 \cos \theta)(-3 \sin \theta) + (1 - 3 \cos \theta - 3 \sin \theta)(3 \sin \theta - 3 \cos \theta) + (3 \sin \theta)(3 \cos \theta) \right] \, d\theta \]
\[ = \int_0^\pi \left[ 9 \cos^2 \theta - 9 \sin^2 \theta - 3 \cos \theta + 3 \sin \theta \right] \, d\theta = 9\pi/2 - 9\pi/2 - 3 \cdot 0 + 3 \cdot 0 = 0. \]

**Problem 4.** In this problem we consider the following two vector fields

\[ \vec{F} = (y \, z \cos (yz) + x, y \cos (yz)) \]
\[ \vec{G} = (x \, z \cos (yz) + x, \, y \cos (yz)) \]

a) (10 points) One of these two vector fields is the gradient of a scalar function and the other is not. Determine which one is which.

**Solution.** We use the "curl test." If the curl of the field is zero, then it is the gradient of a scalar—otherwise it is not. (This test is valid, since the domains of both \( \vec{F} \) and \( \vec{G} \) are all of \( \mathbb{R}^3 \), which is certainly simply connected.) We have

\[ \nabla \times \vec{F} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y \, z \cos (yz) + x & y \cos (yz)
\end{vmatrix} \]
\[ = \vec{i}(\cos(yz) - yz \sin(yz) - \cos(yz) + yz \sin(yz)) - \vec{j}(0 - 0) + \vec{k}(1 - 1) = \vec{0}, \]
so \( \vec{F} \) is the gradient of a scalar function. On the other hand,

\[ \nabla \times \vec{G} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x \, z \cos (yz) + x & y \cos (yz)
\end{vmatrix} \]
\[ = \vec{i}(\cos(yz) - yz \sin(yz) - \cos(yz) + yz \sin(yz)) - \vec{j}(0 - 0) + \vec{k}(1 - 0) \neq \vec{0}, \]
so \( \vec{G} \) is not the gradient of a scalar function.

b) (10 points) Find a scalar function whose gradient is either \( \vec{F} \) or \( \vec{G} \).

**Solution.** We now seek a scalar function, \( f(x, y, z) \) so that \( \nabla f = \vec{F} \). Since \( \frac{\partial f}{\partial x} = y \) we see that \( f(x, y, z) = xy + g(y, z) \). Differentiating both sides of this last equality with respect to \( y \) gives \( z \cos(yz) + x = x + \frac{\partial g}{\partial y} \). From this we see that \( g(x, y) = \sin(yz) + h(z) \) and thus \( f(x, y, z) = xy + \sin(yz) + h(z) \). Differentiating both sides of this new last equation by \( z \) we obtain \( y \cos(yz) = y \cos(yz) + h'(z) \). Thus \( h \) must be a constant, and the general form of the desired \( f \) is

\[ f = xy + \sin(yz) + C \]
where \( C \) is an arbitrary constant.