Math 34
First Midterm

Show your work. Correct answers with no justification may receive little or no credit. No calculators are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space.

Each part of each problem is worth ten points.

Problem 1. Let $\vec{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\vec{f}(x, y) = (x^2 + y^2, xy)$.

a) (10 points) Calculate the matrix derivative $D\vec{f}$.

Solution.

\[
\begin{pmatrix}
2x & 2y \\
y & x
\end{pmatrix}
\]

b) (10 points) Use the chain rule to calculate the matrix derivative of $\vec{f} \circ \vec{f}$. (Your final answer should be a two-by-two matrix of functions of $x$ and $y$.)

Solution.

\[
\begin{pmatrix}
2x & 2y \\
y & x
\end{pmatrix}
\bigg|_{x=x^2+y^2, y=xy} = \begin{pmatrix}
2x^2+y^2 & 2xy \\
xy & x^2+y^2
\end{pmatrix} \begin{pmatrix}
2x & 2y \\
y & x
\end{pmatrix}
\]

\[
= \begin{pmatrix}
4x^3+6xy^2 & 4y^3+6x^2y \\
3x^2y+y^3 & 3xy^2+x^3
\end{pmatrix}
\]
Problem 2. In this problem, we consider a particle moving along the path given by
\[ \vec{x}(t) = (-t \cos t + \sin t, t \sin t + \cos t, t^2). \]

a) (10 points) Find the velocity of the particle.

Solution. The velocity is given by \( \vec{v} = \frac{d}{dt} \vec{x}. \) We get
\[ \vec{v} = (t \sin t, t \cos t, 2t). \]

b) (10 points) Find the acceleration of the particle.

Solution. The acceleration is given by \( \vec{a} = \frac{d}{dt} \vec{v}. \) We get
\[ \vec{a} = (\sin t + t \cos t, \cos t - t \sin t, 2). \]

c) (10 points) Find the function \( s(t) \) which gives the total distance moved by the particle since time 0.

Solution. \( s \) is given by the integral of the speed. In this case the speed has a simple formula: \( \|\vec{v}\| = t \sqrt{5} \), so we get
\[ s(t) = \int_0^t \|\vec{v}(\tau)\| \, d\tau = \sqrt{5} \int_0^t \tau \, d\tau = \frac{\sqrt{5} t^2}{2}. \]

d) (10 points) Suppose that the temperature in the space through which the particle moves is given by \( T = x^3 - 3xy^2 + z \). How fast is the temperature observed by the particle changing when \( t = \pi \)?
Solution. The simplest way to find this rate of change is to use the chain rule. We get
\[ \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt} = (3x^2 - 3y^2)t \sin t + (-6)xyt \cos t + 2t. \]
When \( t = \pi \) we have \( \sin(t) = 0 \) and \( \cos(t) = -1 \), so \( x = \pi \), \( y = -1 \), and \( z = \pi^2 \). Using these values we get \( \frac{dT}{dt} = -6\pi^2 + 2\pi \).

Problem 3. Let \( S \) be the surface defined by \( x^3 + y^2 - z^2 = 1 \).

a) (10 points) Find a parametric expression for the line which intersects \( S \) perpendicularly at \((1,1,1)\).

Solution. The surface \( S \) is a level curve of a function, so we can use the gradient of that function to get a normal vector, \( \vec{n} \), at the desired point. We get
\[ \vec{n} = \nabla(x^3 + y^2 - z^2) \bigg|_{(1,1,1)} = (3x^2, 2y, -2z) \bigg|_{(1,1,1)} = (3, 2, -1). \]
Thus the line we want has vector parametric equation \( \vec{x}(t) = (1, 1, 1) + t\vec{n} \). More explicitly we have
\[ \begin{align*}
x &= 1 + 3t \\
y &= 1 + 2t \\
z &= 1 - 2t.
\end{align*} \]

b) (10 points) Find an equation for the plane tangent to \( S \) at \((1,1,1)\).

Solution. The tangent plane will consist of all of those points, \( P \), that satisfy the condition that the vector from \((1,1,1)\) to \( P \) is perpendicular to \( \vec{n} \). That is to say, in vector terms the desired equation for the coordinates of such \( P \) is
\[ (1 - x, 1 - y, 1 - z) \cdot (3, 2, -2) = 0. \]
Multiplied out, this equation becomes \( 3x + 2y - 2z = 3 \).
Problem 4. In each of the following two parts of this problem, either evaluate the given limit or explain why it does not exist.

a) (10 points)

\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} \]

Solution. When \( y = 0 \), \( \frac{x^2}{x^2 + y^2} = 1 \), no matter what \( x \) is. (Except when \( (x, y) = (0, 0) \) when the expression is undefined.) At the same time, when \( x = 0 \), we get \( \frac{x^2}{x^2 + y^2} = 0 \) no matter what \( y \) is. Thus there is no value to which \( \frac{x^2}{x^2 + y^2} \) approaches as \( (x, y) \) approaches \( (0, 0) \). That is to say, the limit does not exist. \( \square \)

b) (10 points)

\[ \lim_{(x,y) \to (0,0)} \frac{e^x \sin(y)}{x + y + 1} \]

Solution. This expression is made up of continuous functions, by addition and multiplication in such a way that the denominator is nonzero at \( (x, y) = (0, 0) \). Thus, it is continuous at \( (0, 0) \). In other words the limit is just equal to the value of the function evaluated at \( (0, 0) \).

\[ \lim_{(x,y) \to (0,0)} \frac{e^x \sin(y)}{x + y + 1} = \left. \frac{e^x \sin(y)}{x + y + 1} \right|_{(0,0)} = 0. \]

(A cautionary note is in order here: It does not suffice in the second part of this problem to only note that the argument used in the first part fails. For example \( \frac{x^2 y}{x^2 + y^2} \) approaches zero along every straight line through the origin, fails to be continuous at the origin. In particular, its value along the parabola \( y = x^2 \) is the constant, \( \frac{1}{2} \).) \( \square \)