Math 28S
First Midterm

Write all of your work in your bluebooks. You may write the problems in any order you like, but do not put work for more than one problem on the same page of your bluebook. When you are done, number the pages of your bluebook(s) and make a table of contents on the cover of the first one indicating which problems you worked and which pages I can find them on.

Each part of each problem is worth either five or ten points, as marked. (There are 90 points altogether.) For each problem about $n \times n$ matrices or $\mathbb{R}^n$ for general $n$, you can receive $1 - \frac{1}{n}$ of full credit by solving it for a particular value of $n$. (So just doing the $2 \times 2$ case gives half credit, $3 \times 3$ gives $\frac{2}{3}$ credit, and so on.)

Problem 1. In this problem we consider the matrix

$$
\Lambda = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
3 & 3 & 4 & 0 & 0 \\
5 & 5 & 6 & 2 & 1
\end{bmatrix}.
$$

a) (10 points) Use row reduction to find all solutions to

$$
\Lambda \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
$$

Solution. We row reduce the appropriate augmented matrix. I’ve marked the entries about which the pivots are performed.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & | & 1 \\
3 & 3 & 4 & 0 & 0 & | & 2 \\
5 & 5 & 6 & 2 & 1 & | & 3
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & | & 1 \\
0 & 0 & 1 & -3 & 0 & | & -1 \\
0 & 0 & 1 & -3 & 1 & | & -2
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 1 & 0 & 4 & 0 & | & 2 \\
0 & 0 & 1 & -3 & 0 & | & -1 \\
0 & 0 & 0 & 0 & 1 & | & -1
\end{bmatrix}
$$

1
From this we can see that \( x_2 \) and \( x_4 \) are free and that

\[
\begin{align*}
x_1 &= 2 - x_2 - x_4 \\
x_2 &= x_2 \\
x_3 &= -1 + 3x_4 \\
x_4 &= x_4 \\
x_5 &= -1.
\end{align*}
\]

In vector language, this is:

\[
x = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}.
\]

Every solution is of this form where \( x_2 \) and \( x_4 \) are arbitrary scalars. \( \square \)

b) (5 points) Find \( \text{rref}(A) \)

\textbf{Solution.} This is just the unaugmented portion of the row reduced augmented matrix from the previous part:

\[
\begin{bmatrix} 1 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\( \square \)

c) (5 points) What is the rank and nullity of \( A \)?

\textbf{Solution.} The rank is the number of pivot columns—3. The nullity is the number of other columns—2. \( \square \)

d) (10 points) Find a basis for \( \text{im} \; A \).
Solution. A basis for the image is given by the columns of $A$ which correspond to the pivot columns in ref$(A)$. Thus the required basis is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

□

e) (10 points) Find a basis for ker $A$.

Solution. This basis may also be read off of the work in the first part of the column; just let the augmented part of the matrix be zero. Thus the vectors whose coefficients are the free variables in the general solution are a basis for the kernel:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$$

□

Problem 2. In this problem we consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}.$$ 

Let $B$ be the basis whose elements are the columns of $B$.

a) (10 points) Find the inverse of $B$. 

Solution. Again we row reduce the appropriate augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 1 & 0 \\
3 & 3 & 4 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -3 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 2 & -1 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -3 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 5 & -1 & -1 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -3 & 0 & 1
\end{bmatrix}
\]

This gives us

\[
B^{-1} = \begin{bmatrix}
5 & -1 & -1 \\
-1 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix}.
\]

\[\square\]

b) (5 points) Find \(\bar{x}\) if \([\bar{x}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\).

Solution.

\[
\bar{x} = B[\bar{x}]_B = B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}.
\]

\[\square\]

c) (5 points) Find \([\bar{x}]_B\) if \(\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

Solution.

\[
[\bar{x}]_B = B^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}.
\]

\[\square\]
Problem 3. (10 points) Carefully define the terms “linear independence” and “basis.” You may assume that your reader knows what vectors and vector spaces are and is familiar with vector addition and scalar multiplication.

*Solution.* A set of vectors \( \{\tilde{v}_1, \ldots, \tilde{v}_k\} \subseteq V \) in a vector space is linearly independent if the only solution to \( c_1\tilde{v}_1 + \cdots + c_k\tilde{v}_k = 0 \) is \( c_1 = \cdots = c_k = 0 \). That is to say, no nontrivial linear combination of the vectors is zero. Such a set is a basis for \( V \) if in addition to being linearly independent, it also spans \( V \) in the sense that for every \( \tilde{v} \in V \) there is a solution to \( c_1\tilde{v}_1 + \cdots + c_k\tilde{v}_k = \tilde{v} \).

Problem 4. (10 points) True or False: If \( A \) is a matrix and \( A^2 = A \) then \( A \) is diagonal. If false give a specific counterexample. If true, then give a careful proof.

*Solution.* False. A counter example is given by

\[
A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

Problem 5. (10 points) True or False: If \( A \) is a matrix and \( A^2 = 0 \) then \( \text{rank}(A) \leq \text{nullity}(A) \). If false give a specific counterexample. If true, then give a careful proof.

*Solution.* True. If \( A^2 = 0 \), then for every \( \tilde{w} = A\tilde{v} \) we have \( A\tilde{w} = \tilde{0} \). That is to say \( \text{im}(A) \) is a subspace of \( \text{ker}(A) \). Thus

\[
\text{rank}(A) = \dim \text{im}(A) \leq \dim \text{ker}(A) = \text{nullity}(A)
\]
as claimed.