Math 28S
Final Exam

Write all of your work in your bluebooks. You may write the problems in any order you like, but do not put work for more than one problem on the same page of your bluebook. When you are done, number the pages of your bluebook(s) and make a table of contents on the cover of the first one indicating which problems you worked and which pages I can find them on.

Each problem is worth 20 points. (There are 200 points altogether.) For each problem about \( n \times n \) matrices or \( \mathbb{R}^n \) for general \( n \), you can receive \( 1 - \frac{1}{n} \) of full credit by solving it for a particular value of \( n \). (So just doing the \( 2 \times 2 \) case gives half credit, \( 3 \times 3 \) gives \( \frac{2}{3} \) credit, and so on.)

Problem 1 (20 points). Complete each of the following sentences into well-formed, correct definitions. You may assume the reader knows what a vector space is. Each of your definitions should be a single short sentence beginning with my words and ending with yours. The last one should only make sense because of a theorem, which you should state but not prove.

i) A \textit{linear combination} of the elements \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_t \) of a vector space \( V \) is ...  
ii) A subset \( S \) of a vector space \( V \) is \textit{linearly independent} if ...  
iii) A subset \( S \) of a vector space \( V \) is \textit{spans} \( V \) if ...  
iv) A \textit{basis} of a vector space \( V \) is a subset \( S \subseteq V \) such that ...  
v) The \textit{dimension} of a vector space \( V \) is ...  

Solution.

A \textit{linear combination} of the elements \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_t \) of a vector space \( V \) is an expression of the form  

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_t \vec{v}_t. \]

(In the answers below, we will refer to the scalars \( c_1, \ldots, c_t \) as the coefficients of the linear combination.)
A subset $S$ of a vector space $V$ is \textit{linearly independent} if the only linear combinations of the elements of $S$ which evaluate to the zero vector are the ones in which all coefficients are zero.

A subset $S$ of a vector space $V$ is \textit{spans} $V$ if every element of $V$ can be expressed as a linear combination of the elements of $S$.

A \textit{basis} of a vector space $V$ is a subset $S \subseteq V$ such that $S$ spans $V$ and $S$ is linearly independent.

The \textit{dimension} of a vector space $V$ is the number of elements in a basis for $V$. It is a theorem that any two bases have the same number of elements, thus this definition gives a unique number.

\textbf{Problem 2 (20 points).} Let $V$ be the linear subspace of $\mathbb{R}^4$ spanned by the vectors

$$
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
1 \\
1
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}.
$$

Use the Gram-Schmidt process to find an orthonormal basis of $V$.

\textit{Solution.} Following the notation used in our text, we will call the given vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. We will write $\vec{v}_i^\perp$ for a vector in $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_i\}$ which is perpendicular to $\vec{v}_1, \ldots, \vec{v}_{i-1}$. We can find such a vector by subtracting from $\vec{v}_i$ its projection onto the space $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}\}$ or by taking any scalar multiple of this vector. We will write $\vec{e}_i$ for $\vec{v}_i^\perp/|\vec{v}_i^\perp|$. Thus our answer will be $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. Now we proceed to calculation. We have $\vec{v}_1^\perp = \vec{v}_1$ so $\vec{e}_1 = \frac{1}{2} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. For $\vec{v}_2^\perp$ we get the vector

$$
\vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{v}_2 - \frac{3}{4} \vec{v}_1 = \frac{1}{4} \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}.
$$
Since any scalar multiple will do, we take $\vec{v}_2^\perp = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}$. We have $|\vec{v}_2^\perp| = \sqrt{12} = 2\sqrt{3}$

and so $\vec{e}_2 = \frac{\sqrt{3}}{6} \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}$. For $\vec{v}_3^\perp$ we get the vector

$$\vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \frac{\vec{v}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{v}_3 - \frac{2}{12} \vec{v}_2^\perp - \frac{2}{4} \vec{v}_1 = \frac{1}{3} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus we can take $\vec{v}_3^\perp = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{e}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. \[\square\]

**Problem 3** (20 points). Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

i) Find a basis $B$ for $\mathbb{R}^3$ so that the $B$-matrix of $A$ is diagonal.

ii) Find $A^{10}$ using your result to the first part of this problem.

**Solution.** The first step is to find the eigenvalues. To do this we find the roots of the characteristic polynomial. The characteristic polynomial is given by

$$\det (A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2 - \lambda & 1 \\ 1 & 0 & 3 - \lambda \end{bmatrix} = (2 - \lambda) \det \begin{bmatrix} \lambda & -2 \\ 1 & 3 - \lambda \end{bmatrix} = (2 - \lambda)(\lambda^2 - 3\lambda + 2) = -(\lambda - 2)^2(\lambda - 1).$$

Thus the eigenvalues are 2 and 1.
The next step is to find a basis for each eigenspace in each case we proceed by row reduction, using the fact that the kernel of the row reduced echelon form of a matrix is the same as the kernel of the matrix. For the space of eigenvectors with eigenvalue 2 we get

\[
E_2 = \ker \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.
\]

For the space of eigenvectors with eigenvalue 1 we get

\[
E_1 = \ker \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{1}
\]

Putting these two calculations together, we see that an eigenbasis for \( A \) is given by the columns of the matrix \( S = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \). In particular, letting

\[
D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

we have \( S^{-1}AS = D \). Equivalently, \( A = SDS^{-1} \). Thus \( A^{10} = SD^{10}S^{-1} = \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} S^{-1} \).

To finish we need to calculate \( S^{-1} \), which we do by means of another row reduction.

\[
\begin{bmatrix} 0 & -1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix}.
\]
So we have

\[
A^{10} = \mathbf{SD}^{10} \mathbf{S}^{-1} = \begin{bmatrix}
0 & -1 & -2 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix} \begin{bmatrix}
2^{10} & 0 & 0 \\
0 & 2^{10} & 0 \\
0 & 0 & 1^{10} \\
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 2 \\
-1 & 0 & -1 \\
\end{bmatrix} = \begin{bmatrix}
-2^{10} + 2 & 0 & -2 \cdot 2^{10} + 2 \\
2^{10} - 1 & 2^{10} & 2^{10} - 1 \\
2^{10} - 1 & 0 & 2 \cdot 2^{10} - 1 \\
\end{bmatrix} = \begin{bmatrix}
-1022 & 0 & -2046 \\
1023 & 1024 & 1023 \\
1023 & 0 & 2047 \\
\end{bmatrix}.
\]

\[\square\]

Problem 4 (20 points). Row reduce \(A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3 \\
\end{bmatrix}\). Use your result to help you find bases for the image and the kernel of \(A\).

**Solution.** We will use the fact that the kernel of \(A\) is the same as the kernel of its reduced row echelon form, and that the image of \(A\) is spanned by the columns of \(A\) which are in the positions of the pivot columns of its reduced row echelon form.

First we row reduce:

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 2 \\
0 & -1 & -1 \\
0 & -1 & -1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

Thus, we see that \(\ker A = \text{Span} \left\{ \begin{bmatrix}
-1 \\
-1 \\
1 \\
\end{bmatrix} \right\} \) and \(\text{im } A = \text{Span} \left\{ \begin{bmatrix}
1 \\
0 \\
2 \\
1 \\
\end{bmatrix} \right\} \). \[\square\]

Problem 5 (20 points). Suppose that \(A\) is a real symmetric matrix and that \(\bar{v}_1\) and \(\bar{v}_2\) are eigenvectors with different eigenvalues. Give a carefully written but short proof that \(\bar{v}_1\) is perpendicular to \(\bar{v}_2\).

**Solution.** By hypothesis, we have \(A\bar{v}_1 = \lambda_1 \bar{v}_1\) and \(A\bar{v}_2 = \lambda_2 \bar{v}_2\) with \(\lambda_1 \neq \lambda_2\). Thus we have

\[
\lambda_1 (\bar{v}_1 \cdot \bar{v}_2) = (\lambda_1 \bar{v}_1)^T \bar{v}_2 = (A\bar{v}_1)^T \bar{v}_2 = \bar{v}_1^T A^T \bar{v}_2 = \bar{v}_1 A \bar{v}_2 = \lambda_2 (\bar{v}_1 \cdot \bar{v}_2).
\]

It follows that \((\lambda_1 - \lambda_2)(\bar{v}_1 \cdot \bar{v}_2) = 0\). Thus—because \(\lambda_1 \neq \lambda_2\)—we have \(\bar{v}_1 \cdot \bar{v}_2 = 0\). \[\square\]
The rest of the problems are true or false. For each statement, determine whether it is true or is false. Then either give an example or a proof justifying your assertion.

Problem 6 (20 points). There is an invertible matrix, $S$ so that $S^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} S$ is diagonal. (For extra credit state and prove a generalization of your assertion which is valid for $n \times n$ matrices.)

Solution. False. Let $A$ be the $n \times n$ matrix which has 1 for every diagonal entry and for every entry just above the diagonal, but is zero everywhere else. That is to say,

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} = I_n + J_n$$

where $I_n$ is the identity matrix and $J_n = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}$ is the matrix which is ones just above the diagonal and zero everywhere else. Now, if $S^{-1}AS$ where diagonal, then the columns of $S$ would be an eigenbasis for $A$. Since the characteristic polynomial of $A$ is $(1 - \lambda)^n$, the only eigenvalue of $A$ is $n$, and the corresponding eigenspace is $E_1 = \ker J$ which is one-dimensional, since the rank of $J$ is $n - 1$. Thus as long as $n > 1$ there is no eigenbasis for $A$ so there is no $S$ as described in the problem. □

Problem 7 (20 points). There is a $3 \times 4$ matrix with no zero entries with nullity equal to 1.

Solution. True. By the rank plus nullity theorem, we simply need the matrix to have rank three. This will be the case if the first three columns are linearly
independent. One example is

\[
\begin{bmatrix}
1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
\end{bmatrix}
\]

\[\Box\]

Problem 8 (20 points). There is a $4 \times 4$ real symmetric matrix with no real eigenvalues.

Solution. True. By the spectral theorem, every real symmetric matrix is diagonalizable over the reals and in particular must have at least one real eigenvalue. (In fact, it has $n$ if they are counted with their multiplicities.) \[\Box\]

Problem 9 (20 points). There is a $4 \times 4$ real matrix with no real eigenvalues.

Solution. True. The matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

has characteristic polynomial $(\lambda^2 + 1)^2$ which has no real roots and so has no real eigenvalues. \[\Box\]

Problem 10 (20 points). If all the diagonal entries of a square matrix are odd integers and all the other entries of the matrix are even integers, then the matrix is invertible.

Solution. True. For the concreteness' sake, assume the matrix in question is $n \times n$. The determinant of such a matrix is an integer, by any of our formulas for the determinant. Furthermore, only one of the $n!$ entries in the full expansion of the determinant (the one consisting of the product of the diagonal entries) is odd. Thus, the determinant is an odd integer. In particular, the determinant cannot be zero, so the matrix must be invertible. \[\Box\]

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