FIN DE CALCUL: NOTES FOR MATH 26

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ABSTRACT. These notes are intended to be used for the second half of Math 26 at Swarthmore College. There is little in these notes which is not distilled fairly directly from the references listed at the end.

In particular, the material on differential equations is mostly distilled from Chapters 8 and 9 of the excellent text, [Ap67], by Tom Apostol. I've left out most of the proofs, extracted the simpler parts, added one or two more straightforward examples. The interested reader should consult [Ap67] for an unbowdlerized version.

The material on the complex numbers draws on the treatment at the beginning of [Car69]. Here, I've added (fairly standard) material in an effort at making the treatment more accessible to the intended audience—mostly first year students at Swarthmore College, many who will use calculus in the future, and a mix of possible math majors and students who are not contemplating a mathematics major.

Many of the examples and exercises have been taken from exams previously given in the course. The reader should be cautioned that future exams are not likely to include problems which are worked out explicitly in these notes.

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0. Preface

Many times what makes an ending a good ending is that it is a good beginning. I hope that this was true for each student in Math 26 as she celebrated "commencement" at the end of the schooling that prepared her for Swarthmore. I also hope that it will be true of the commencement each student will celebrate at the end of their Swarthmore studies.

My aim in writing these notes is to provide such a good ending to the study of basic single variable calculus.

I have tried to make no assumptions about the reader, other than that she has recently finished the study of sequences and series (of real numbers) and is finishing up her study of single variable calculus. Some readers may never take another mathematics course—although I expect that most every student who has taken this much calculus will use some of it in other courses or in their work after college. Other readers may be beginning their studies as mathematics majors, and will be taking many more mathematics courses.

The specific aims of this part of the course are the following: To give students the ability to solve simple ordinary differential equations, including separable equations, first order equations, and second order equations with constant coefficients. To give several extended examples of the application of these techniques. To introduce the complex numbers including a sketch of some foundational issues. To discuss the geometry of the complex numbers and of basic complex arithmetic. To add complex sequences and series to the students' existing understanding of sequences and series. To give a series-based development of the complex exponential function and to show how the trigonometric functions and their properties follow from the complex exponential function. To revisit the students' existing understanding of differential equations highlighting the simplification offered by complex exponential functions. To revisit the students' existing understanding of Fourier series highlighting the simplification offered by complex exponential functions.
There are also important aims of these notes which are more vague and not captured by a mere listing of the topics to be covered. I want to show off a little bit of the nature of mathematics and give pointers in the direction of material that is covered in other courses. I want to show by example how mathematical thought and ideas can encompass arguments or discussions that are longer than just a short exercise. Metaphorically put, up to this point most students have been taught in calculus how to write the equivalent of paragraphs. In these notes I hope to give some examples of short stories. Students who go on in mathematics will have to learn both to read and (still later) to write in such longer units. I hope that a glimpse of what longer mathematical tales look like will be enjoyable and instructive for all students in the course.

There are a few things that I feel should be explicitly asserted as not being aims of this course. This is not a technical course. While techniques of problem solving play a substantial role, we aim to place those techniques in a bigger picture with the aim of giving a glimpse of some of the interesting mathematics which can come after calculus. On the other hand, while many parts of this course are about theory, this is not a theory course. In particular, many proofs will be omitted and others may only be sketched. We will include proofs or sketches only when we think that the resulting discussion is particularly illuminating or beautiful. This course is not intended to teach how to create proofs, but rather to teach by example how to work through a few mathematical stories of various lengths. I aim to show by example a little bit of what mathematical theory is and why one might want to study it.

Finally, while we discuss differential equations and complex numbers in this course (and to a lesser extent algebra and analysis), this is not a substitute course in any of those subjects. We have tried to carefully avoid stealing topics from our courses in those subjects. We hope students will find the techniques and ideas in this course useful, enjoyable, and interesting. Students who do are encouraged to consider taking one or more of the courses which develop the material in detail:

- Math 44: Differential Equations
- Math 63: Introduction to Real Analysis
- Math 67: Introduction to Modern Algebra
- Math 103: Complex Analysis
1. Review of integration by parts

In this section we give a very brief review of integration by parts. This review should provide the reader who has not seen integration by parts in her previous study of calculus with more than enough to read the rest of these notes. What we aim for is a basic understanding of integration by parts and an ability to do the easy examples with no fuss or bother. Interestingly, the most traditional more complicated examples of integration by parts may be avoided altogether by use of the complex methods given later in these notes.

1.1. Review of integration. In this subsection we review the notation and ideas of (indefinite) integration.

As usual, we will write

$$\int f(x) \, dx = F(x) + c$$

(1)

to indicate that $\frac{d}{dx} F(x) = f(x)$. Furthermore, in expressions involving integrals, if $u$ stands for a function of $x$ we will write $du$ as shorthand for $\frac{du}{dx} \, dx$. In this shorthand, if we write $u = F(x)$, Equation (1) then becomes

$$\int du = u + c.$$

In such expressions the entity, $du = f(x) \, dx$, is neither a number or a function—instead it is shorthand for something which may be used as part of an integral expression. We call $du$ a differential form.

We write $f(x) \, dx = g(x) \, dx$ when $f(x) = g(x)$. Thus, $du = dv$ when $\frac{du}{dx} = \frac{dv}{dx}$. In this case, we have $\int du = \int dv$, but of course, $u$ and $v$ may differ by a constant.

Example 1.1. Here is a very simple example. To find $\int \cos(3x) \, dx$ we could set $u = 3x$. This would give $du = 3 \, dx$, so we then would have

$$\int \cos(3x) \, dx = \int (\cos u) \frac{1}{3} \, du = \frac{1}{3} \sin u + c = \frac{1}{3} \sin(3x) + c.$$

In this case, and in many cases of simple substitution, we can shorten our description of the solution by not introducing the letter $u$. The result is a shorter one line solution:

$$\int \cos(3x) \, dx = \int \cos(3x) \frac{1}{3} \, d(3x) = \frac{1}{3} \sin(3x) + c.$$

The well seasoned integrator should probably not even make the substitution so explicit—she could guess that $\sin 3x$ is close to the right answer and reason
in the following way:

\[ \int \cos(3x) \, dx = \int \frac{1}{3} d\sin(3x) = \frac{1}{3} \sin(3x) + C. \]

Even though this example is easy enough that the reasoning is pretty clear, let me offer a narration of this last expression of the solution which will point the way toward solving more difficult problems: Faced with the original problem, we “guess\(^1\)” the answer \(\sin(3x)\). We check \(d(\sin(3x)) = 3\cos(3x) \, dx\). This is close to, but not exactly the original differential we were faced with. Dividing by 3 gives \(\frac{1}{3} d(\sin(3x)) = \cos(3x) \, dx\) which after integration is what we were after. I call this method of integration “guess, check and correct\(^2\).” All substitutions can be expressed this way and we will see below that integration by parts may also be expressed as “guess, check, and correct” calculations.

These three arguments are really just three ways of saying pretty much the same thing. Which you prefer is up to you, but you should be able to understand all three.

Multiplication of a differential form by a function and addition of another differential form to another are given by the following rules.\(^3\)

\begin{align*}
(2) & \quad f(x) \, dx + g(x) \, dx = (f(x) + g(x)) \, dx \\
(3) & \quad f(x)(g(x) \, dx = (f(x)g(x)) \, dx
\end{align*}

With these rules for the algebraic manipulation of differential forms, the usual most basic rules for differentiation can be given as follows, where \(c\) is

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\(^1\)We aren’t really guessing that this is the answer, but rather that it is close to the answer.

\(^2\)Actually, many call this method “guess and fudge.” However the term fudge often connotes falsification, where here all we are trying to do is fix a wrong—but close—answer to make it exactly correct.

\(^3\)It is, perhaps, natural for a student who has already had a year of calculus to feel that the rules stated in the formulas at (2) and (3) are “obvious.” However, since differential forms are neither numbers nor functions, the rules for their manipulation do not follow from the algebra of numbers and functions. Here, we give just enough information for the student to perform calculations correctly. Our rules are set up so that whenever we have an equality of differential forms we may integrate both sides to get an equality of functions. A student who wishes a more thorough understanding of algebraic systems in general should take an abstract algebra course, such as Swarthmore’s Math 67. A student who wishes a more thorough understanding of the algebra of differential forms can get it in Math 101 (or sometimes in Math 35).
a constant.

\[(4) \quad d(cu) = c \, du \]
\[(5) \quad d(u + v) = du + dv \]
\[(6) \quad d(uv) = u \, dv + v \, du \]

When we integrate both sides of Equation (4) we get

\[ \int c \, du = \int d(cu) = cu + C = c \int du \]

which, when \( du = f(x) \, dx \), is the familiar rule

\[ \int cf(x) \, dx = c \int f(x) \, dx. \]

Similarly, when we integrate Equation (5) we get

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx. \]

Integration by parts is what we get when we integrate both sides of Equation (6). In the next subsection, we will elaborate on this observation.

1.2. Integration by parts. Formally, integration by parts is what we get when we integrate both sides of the product rule as presented in Equation (6). This yields

\[(7) \quad \int u \, dv + \int v \, du = \int d(uv) = uv + C, \]

which is usually written in the following form.

\[ \int u \, dv = uv - \int v \, du \]

Just as with substitution, one may use formula (8) in a formulaic manner, explicitly writing out \( u \) and \( v \), or one can use the “guess, check, and correct” style. We will work a couple of examples both ways. As with substitution, the two ways are equivalent. You may choose either one.

**Example 1.2.** We perform the integral \( \int xe^x \, dx \) by parts giving a full narration of the “guess, check and correct” style of solution. We are looking for an \( F(x) \) so that \( dF = xe^x \, dx \). Start by “guessing” \( xe^x \). We know that this is
wrong, since the product rule gives $d(xe^x) = xe^x \, dx + e^x \, dx$ which includes an extra term. We can fix this by subtracting $e^x \, dx$ from both sides. Thus,

$$xe^x \, dx = d(xe^x) - e^x \, dx = d(xe^x) - d(e^x) = d(xe^x - e^x),$$

so

$$\int xe^x \, dx = \int d(xe^x - e^x) = xe^x - e^x + C.$$

**Example 1.3.** We perform the integral $\int xe^x \, dx$ by parts using the “guess, check and correct” style of solution, but without the full narration. Here we would just write,

$$\int xe^x \, dx = \int d(xe^x) - \int e^x \, dx = xe^x - e^x + C.$$

With the information that we are integrating by parts, the reader should be able to provide the narration for herself.

**Example 1.4.** Finally we perform the integral $\int xe^x \, dx$ by parts using the explicit formulaic style. Here we must find $u$ and $v$ so that $xe^x \, dx = u \, dv$ and so that $v \, du$ is easier to integrate. We take $u = x$ and $v = e^x$. Then $du = dx$ and $dv = e^x \, dx$. Applying formula (8) we get

$$\int xe^x \, dx = \int u \, dv = uv - \int v \, du = xe^x - \int e^x \, dx = xe^x - e^x + C.$$

Beginning partial integrators naturally often prefer the explicit formulaic method.

1.3. **Definite integrals.** As you learned before taking this course, if we have

$$\int f(x) \, dx = F(x) + C$$

then the definite integral from $a$ to $b$ is given by

$$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$$

where the fundamental theorem of calculus tells us that this quantity is the area under $y = f(x)$ between $x = a$ and $x = b$. In the case of integration by parts, this becomes

$$\int_{x=a}^{x=b} u \, dv = [uv]_{x=a}^{x=b} - \int_{x=a}^{x=b} v \, du.$$

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*In fact, we chose $xe^x$ from all possible guesses exactly because one of the terms in its derivative (via the product rule) was what we were looking for, and the other one was simpler than what we started with.*
There is a very nice geometric picture of this definite integral formula, which I learned from my favorite old calculus book, [Cou88]. Figure 1 gives a version of Courant’s diagram. Here the area of the large rectangle is \( u(b)v(b) \) and each of the three regions it is divided into is labeled with its own area. Thus the fact that they fill the big rectangle entirely displays the product rule in the form

\[
\int_{x=a}^{x=b} u \, dv + \int_{x=a}^{x=b} v \, du.
\]

1.4. Exercises.

Exercise 1.1. Use integration by parts in whichever style you like to perform the following integrals.

a) \( \int x \sin(x) \, dx \).

b) \( \int x \cos(x) \, dx \).
c) \( \int x \ln x \, dx \).
d) \( \int x^2 \ln x \, dx \).
e) \( \int \ln x \, dx \).
f) \( \int_0^1 xe^x \, dx \).

2. Separation of Variables

In this section we give a few simple examples of separation of variables to illustrate the technique. The student should consult her main text for a more detailed discussion—if more details are necessary or desirable.

**Example 2.1.** We find a solution to the differential equation

\[ xy' - (2 + x)y = 0 \]

which satisfies \( y(1) = 1 \). First we use separation of variables to find a general solution:

\[
\frac{dx}{x} = \frac{dy}{(2 + x)y} = \left( \frac{2}{x} + 1 \right) \, dy
\]

\[
\ln |x| = 2 \ln |x| + x + C
\]

\[
y = A(x^2 e^x).
\]

Here \( C \) and \( A \) are constants of integration. Next we find a particular solution for which \( y(1) = 1 \). Plugging \( x = 1 \) into our general solution, we see that \( 1 = A e \), so \( A = \frac{1}{e} \) and our solution to the initial value problem is \( y = x^2 e^{x-1} \).

(One should differentiate this and check that it solves our equation.) Figure 2 displays the graphs of several solutions of Equation (9). This graph suggests that \( y' = 0 \) when \( x = 2 \). In fact, taking \( x = 2 \) in Equation (9) proves that this is the case. Similarly, it appears that both \( y \) and \( y' \) vanish when \( x = 0 \). We may verify these observations hold for the solutions obtained by setting

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5We chose the condition \( y(1) = 1 \) out of the blue to illustrate the technique—we could have picked any other initial or boundary value condition to satisfy instead.

6Perhaps a little explanation is required to know how to get to the last line from the penultimate line. Generally speaking, if \( \ln |y| = f(x) + C \) where \( C \) is an arbitrary constant of integration, then applying the exponential function to both sides gives \( |y| = e^C e^{f(x)} \), so \( y = \pm e^C e^{f(x)} \). If \( C \) is an arbitrary real constant, then \( e^C \) is an arbitrary positive constant, so writing \( \Lambda = \pm e^C \), we get \( y = \Lambda e^{f(x)} \) where \( \Lambda \) is an arbitrary non-zero constant. In our example above, we can see that \( \Lambda = 0 \) also solves our problem, although the separation isn’t valid when \( y = 0 \).
\( x = 0 \) in Equation (10) and in the formula for \( y' \) obtained by differentiating both sides of Equation (10).

**Example 2.2.** We consider the initial value problem

\[
\frac{dy}{dt} = y(10 - y) \quad y(0) = 1.
\]

First we find the general solution. In the process, we will need the formula

\[
\frac{1}{y(10 - y)} = \frac{1}{10} \frac{1}{y} + \frac{1}{10 - y}
\]

which may be obtained by solving

\[
\frac{1}{y(10 - y)} = \frac{A}{y} + \frac{B}{10 - y}
\]
Figure 3. Several solutions to $y' = y(10 - y)$

for $A$ and $B$.

\[
\int \frac{dy}{y(10 - y)} = \int dt \\
\int \left( \frac{1/10}{y} + \frac{1/10}{10 - y} \right) = t + C \\
\frac{1}{10} \ln |y| - \frac{1}{10} \ln |10 - y| = t + C \\
\ln \left| \frac{y}{10 - y} \right| = 10t + 10C \\
\frac{y}{10 - y} = Ae^{10t}.
\]

Plugging the initial condition in to this gives $A = \frac{1}{9}$ so the solution to the initial value problem is given by $\frac{y}{10 - y} = \frac{1}{9}e^{10t}$. If we solve this for $y$ we get

\[
y = \frac{10}{9e^{-10t} + 1}.
\]

Figure 3 displays the graphs of several solutions with various initial con-
ditions. The graph suggests that \( y = 0 \) and \( y = 10 \) are constant solutions. (Such solutions are also called equilibrium solutions.) Indeed, we can see this analytically by setting \( y' = 0 \) in the differential equation yielding \( y(10 - y) = 0 \) which has \( y = 0 \) and \( y = 10 \) as solutions.

2.1. Exercises.

Exercise 2.1. Consider the differential equation

\[
\frac{dy}{dt} = a - by
\]

where \( a \) and \( b \) are non-zero constants.

a) Find the general solution, using separation of variables.
b) Find the particular solution for which \( y = 0 \) when \( t = 0 \).
c) Are there any equilibrium solutions? If so, find them.
d) Show that when \( b > 0 \) all solutions become close to the constant \( y = \frac{a}{b} \) when \( t \) is very large and positive.

3. First Order Linear Ordinary Differential Equations

In this section we will explain what a first order linear ordinary differential equation is, we will present the method of integrating factors, and we will work through a few exercises. First order linear ordinary differential equations are general enough that they model many phenomena. As we will see in this section, such differential equations are also simple enough that their solution always can be reduced to integration.

3.1. The definition. A first order linear ordinary differential equation is one of the following form.

\[
\frac{dy}{dx} + y p(x) = q(x)
\]

In this equation, we take \( p(x) \) and \( q(x) \) to be functions which have been specified in advance and seek to find a function \( y = f(x) \) which satisfies the stated equality. The domain of definition of \( p, q, \) and \( f \) may be all of the real numbers or a specified subset—usually an interval.

Equation (11) is called “first order” because the highest derivative of \( y \) which occurs is the first derivative, \( \frac{dy}{dx} \). It is called “linear” because \( y \) and \( \frac{dy}{dx} \) appear in their respective terms multiplied only by a function of \( x \) and not by \( y \) or a derivative of \( y \). It is “ordinary” because it involves only ordinary differentiation—not partial differentiation.
Examples 3.1.

1. \( \frac{du}{dx} + x^2y = \cos(x) \) is a first order linear ordinary differential equation.
2. \( y \frac{du}{dx} = \cos(x) \) is a first order ordinary differential equation which is not linear.
3. \( \frac{d^2u}{dx^2} + x \frac{du}{dx} + x^2y = \cos(x) \) is a second order linear ordinary differential equation—it is not first order.
4. \( \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \) is a second order partial differential equation—it is not ordinary.

In what follows, we will not give careful proofs of the sort that are found in more advanced or more theoretical math classes. We will not even state a careful theorem. However if we were to wish for more rigor, all we would need for our hypotheses is that \( p(x) \) and \( q(x) \) are continuous\(^7\) for \( a \leq x \leq b \) where \([a, b]\) is an interval in which we wish to solve for \( y \) as a function of \( x \). It would then follow that there is a general solution \( y = f_{gen}(x) \) which depends on a constant which we may use to specify the value \( y \) at any one particular point in the interval, \([a, b]\). That is to say, if we are given \( x_0 \) in \([a, b]\) and we take any value \( y_0 \), then by choosing the constant appropriately in \( f_{gen} \) we obtain a function \( y = f(x) \) which satisfies Equation (11) and also satisfies \( f(x_0) = y_0 \). We call the condition \( y_0 = f(x_0) \) a "boundary condition" or an "initial value condition" and we call the \( f(x) \) which we obtained a "particular solution" to Equation (11). The following example should help to clarify this language.

Example 3.2. Consider the differential equation

\[
\frac{du}{dx} + y \tan x = 0.
\]

This is a first order linear differential equation. The coefficient \( \tan(x) \) is continuous on the interval \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). We will look for solutions on that interval. In this case the method of separation of variables suffices. The separated version of the equation is \( \frac{du}{y} = -\tan x \, dx \). Integrating both sides gives

\[
\ln |y| = \ln |\cos x| + C.
\]

Exponentiating both sides gives the general solution

\[
y = A \cos x
\]

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\(^7\) The careful reader of the following examples will notice that even these fairly weak conditions need not be satisfied in order for a solution to exist. The conditions are, however, sufficient and easy to state so we will leave this line of inquiry unexplored. The curious reader can consult [Ap67] or [BD65].
where $A$ is an arbitrary constant. We can (and should) check our work by plugging back into the original equation. We do this check by taking $y = A \cos x$ and noting that we have $\frac{dy}{dx} + y \tan x = -A \sin x + A \cos x \tan x = 0$

In Figure 4 we show the graph of $y = A \cos x$ over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ for various values of the constant $A$. Finally, we may set the constant, $A$, in this general solution, $y = A \cos(x)$, to produce a particular solution satisfying a condition. For example, suppose we wish to find a solution to the equation for which $y = \frac{1}{2}$ when $x = 0$. Putting $x = 0$ in the general solution gives $y = A$. Therefore, the desired particular solution is $y = \frac{1}{2} \cos x$.

In the following two remarks we note that Equation (11) is very easy to solve if either $p(x)$ or $q(x)$ is the constant zero function.

**Remark 3.3.** The equation (12) we solved in Example 3.2 was an instance of the general Equation (11) in which $q(x) = 0$. As we will now see, we may deal with any equation like this exactly as we did in the example. The equation

$$\frac{dy}{dx} + yp(x) = 0$$
is separable with separated form \( \frac{du}{y} = -p(x) \). If \( P(x) \) is any antiderivative of \( p(x) \), then integrating both sides gives

\[
\ln |y| = -P(x) + C.
\]

Exponentiating both sides of this equation gives

\[
y = Ae^{-P(x)}
\]

where \( A \) is an arbitrary constant. If we want \( y = y_0 \) when \( x = x_0 \) in this general solution, we need \( y_0 = Ae^{-P(x_0)} \). We may arrange this by choosing \( A = y_0e^{P(x_0)} \). Thus, the desired particular solution is given by

\[
y = y_0e^{P(x_0)}e^{-P(x)} = y_0e^{P(x_0) - P(x)}.
\]

**Remark 3.4.** Instances of Equation (11) in which \( p(x) = 0 \) are even easier to deal with. In this case, the equation becomes

\[
(13) \quad \frac{dy}{dx} = q(x).
\]

Solutions of Equation (13) are just antiderivatives of \( q(x) \). Thus if \( y = Q(x) \) is any such antiderivative, then the general solution is \( y = Q(x) + C \). If we want \( y = y_0 \) when \( x = x_0 \) in this general solution, we need \( y_0 = Q(x_0) + C \), which we may arrange for by taking \( C = y_0 - Q(x_0) \). Thus the desired particular solution is

\[
y = Q(x) + y_0 - Q(x_0).
\]

The following example shows that separation of variables may not suffice in situations where neither \( p(x) \) nor \( q(x) \) is the constant zero function.

**Example 3.5.** Consider the differential equation

\[
(14) \quad \frac{dy}{dx} + y\tan x = 1.
\]

This is, again, a first order linear differential equation. In this case, we need a new method of solution, since we cannot separate the variables. In Section 3.2, we will give a method resulting in a general formula solving such equations. We could just cite those results here. Instead we will puzzle this example out as a means of showing where the trick lying behind the method of Section 3.2 comes from.

As a starting point, let's think a little more about Example 3.2. The solution we obtained there, \( y = A\cos x \), can be written \( y\sec x = A \) or
\[ \frac{dy}{dx} (y \sec x) = 0. \] In fact, for any \( y \), the product rule gives

\[ \frac{d}{dx} (y \sec x) = \frac{dy}{dx} \sec x + y \tan x \sec x. \] (15)

We can now see another way we could have argued our solution to Example 3.2: Multiply the Equation (12) through by \( \sec x \) to obtain

\[ \frac{dy}{dx} \sec x + y \tan x \sec x = 0, \]
recognize the left hand side as the derivative of the product, giving

\[ \frac{d}{dx} (y \sec x) = 0, \]
and deduce that \( y \sec x \) is constant—arriving at the solution without separating variables. This method is a little more complicated, and a little trickier, but it has the advantage that it can work even when the equation at hand is not separable.

Returning to the problem at hand, we can apply the same technique to Equation (14) since its left hand side is the same as the left hand side of Equation (12) from Example 3.2. Multiplying both sides of Equation (14) by \( \sec x \) and recognizing the left hand side of the result as a derivative of a product, we get

\[ \frac{dy}{dx} \sec x + y \tan x \sec x = \sec x \]

\[ \frac{d}{dx} (y \sec x) = \sec x \]

or \( d(y \sec x) = (\sec x)dx \). Integrating both sides (using an integral table for the right hand side) we get

\[ y \sec x = \frac{1}{2} \ln \left| \frac{(\sin x) + 1}{(\sin x) - 1} \right| + C. \]

Equivalently,

\[ y = \frac{\cos x}{2} \ln \left| \frac{(\sin x) + 1}{(\sin x) - 1} \right| + C \cos x. \] (16)

As before, given initial conditions, we can solve for \( C \). In Figure 5 we show the graph of the \( y \) given in Equation (16) over the interval \(( -\frac{\pi}{2}, \frac{\pi}{2} )\) for various values of the constant \( C \).
3.2. Integrating factors. We now return to the general first order linear ordinary differential equation (11) which we repeat here for reference.

\[ \frac{dy}{dx} + yp(x) = q(x) \]

We wish to use the kind of trick we did in Example 3.5. That is to say, we wish to multiply both sides of Equation (17) by a term \( r(x) \) so that the left hand side is the derivative of a product. That is to say, we want

\[ \frac{dy}{dx} r(x) + yp(x)r(x) = \frac{ds(x)}{dx} \]

Expanding the right hand side of Equation (18), we see that what we want is

\[ \frac{dy}{dx} r(x) + yp(x)r(x) = \frac{dy}{dx}s(x) + y \frac{ds(x)}{dx} \]

This could be accomplished with \( r(x) = s(x) \) as long as we can arrange for

\[ \frac{dr}{dx} = p(x)r(x) \]

But this equation (19) is separable! Separated, it is \( \frac{dr}{r} = p(x) \, dx \). This gives \( \ln |r(x)| = P(x) + C \) where \( P(x) \) is any antiderivative of \( p(x) \). Thus, \( r(x) = Ae^{P(x)} \) solves Equation (19). We only needed one solution, so we will
take \( r(x) = e^{p(x)} \). We’ve now found a term to multiply by so that the left hand side of Equation (17) becomes the derivative of a product. This term is often called an integrating factor.

If \( P(x) \) is any antiderivative of \( p(x) \), then multiplying both sides of Equation (17) by \( r(x) = e^{P(x)} \) gives

\[
\frac{du}{dx}e^{P(x)} + yp(x)e^{P(x)} = q(x)e^{P(x)}.
\]

Recognizing the left hand side of equation (20) as the derivative of a product, we get

\[
\frac{d}{dx}(ye^{P(x)}) = q(x)e^{P(x)}.
\]

It follows that

\[
ye^{P(x)} = \int q(x)e^{P(x)} \, dx.
\]

That is to say, the general solution to Equation (17) is given by

\[
y = e^{-P(x)}\int q(x)e^{P(x)} \, dx
\]

where the indefinite integral is to be interpreted as a general antiderivative. As we have mentioned, it is traditional to call \( e^{P(x)} \) an integrating factor.

Remark 3.6. The formula presented as Equation (21) is harder to remember than is the form of the integrating factor, \( r(x) = e^{\int p(x) \, dx} = e^{P(x)} \), together with the idea of multiplying through by the integrating factor. Often, just remembering that we wish to multiply by a term turning the left-hand side into the derivative of a product is all that is necessary to solve the general first order linear ordinary differential equation.

Example 3.7. We find the general solution to

\[
\frac{dy}{dx} = y + x.
\]

We take \( p(x) = -1 \) and \( q(x) = x \) in Equation (17), so \( P(x) = -x \) and we have

\[
ye^{-x} = \int xe^{-x} \, dx = -xe^{-x} - e^{-x} + C
\]

so

\[
y = -x - 1 + Ce^{x}.
\]

Here we can relate \( C \) to \( y(0) \) by just evaluating everything at 0. \( y(0) = C - 1 \), so \( C = y(0) + 1 \). Notice that there are no equilibrium solutions—if there were
an equilibrium solution, then setting $y'$ to zero in Equation (22) would give $y = -x$ which is not a constant. Notice also that for $x$ large and negative the exponential term in the general solution is very small. Thus, all solutions are well approximated by the line $y = -x - 1$ when $x$ is large and negative. We can see this behavior quite clearly in Figure 6.

*Example 3.8.* Consider the differential equation

$$\frac{dy}{dt} = 2e^{-t} - 3y.$$

We can find the general solution using an integrating factor. First, we rewrite the equation so that the part not involving $y$ is on the right: $\frac{dy}{dt} + 3y = 2e^{-t}$. Now, either by inspection—looking for the product rule—or by remembering the formula, we see that $e^{3t}$ is a good integrating factor. Multiplying by this
and recognizing the product rule, we get
\[ e^{3t} \frac{dy}{dt} + 3e^{3t}y = e^{3t}2e^{-t} \]
\[ \frac{d}{dt} (e^{3t}y) = 2e^{2t} \]
\[ e^{3t}y = e^{2t} + C \]
\[ y = e^{-t} + Ce^{-3t} \]

Figure 7 displays this general solution for several values of the parameter C. We can find the particular solution for which \( y = 0 \) when \( t = 0 \) by setting the constant, \( C \), to the appropriate value. To find this value we plugging the desired \( y \) and \( t \) values into the general solution. This gives \( 0 = 1 + C \), so \( C = -1 \). Thus the solution we seek is
\[ y = e^{-t} - e^{-3t} \]

Notice that for \( t \) large the exponential terms in this solution are very small. Thus all solutions become very close to zero as \( t \) is large and positive. We
can see this behavior clearly in Figure 7. Finally, note that we can see that there are no equilibrium solutions. In fact, if \( y \) is a constant solution, then the original equation gives \( 0 = 2e^{-t} - 3y \) which contradicts the assumption that \( y \) is constant.

Here are some problems to practice with. In each case either find the general solution, or if an initial condition is specified find the solution satisfying the initial equation. Solve the equations using an integrating factor even if they are separable.

3.3. Exercises.

**Exercise 3.1.** \( y' - 3y = e^{2x} \).

**Exercise 3.2.** \( y' + y = e^{-x} \) and \( y = 0 \) when \( x = 0 \).

**Exercise 3.3.** \( y' = 100 - y \).

**Exercise 3.4.** \( xy' + (1 - x)y = e^{2x} \).

**Exercise 3.5.** \( xy' + (1 - x)y = \frac{e^{2x}}{x} \).

**Exercise 3.6.** Use an integrating factor to find the general solution to Equation (12). (You should get the same answer as we did when we used the method of separation of variables.) Both methods are fine to use. Which do you like better?

**Exercise 3.7.** Use an integrating factor to solve the differential equation

\[
\frac{dy}{dt} = a - by
\]

where \( a \) and \( b \) are non-zero constants.

4. **Newton's Law of Cooling**

In this section we consider an extended example—Newton's law of cooling. The differential equations considered here also model many other phenomena such as diffusion and other processes in which difference drives change.

Consider a small uniform object being heated or cooled in a larger environment. For concreteness' sake, we may imagine a yam in an oven. Let the temperature of the yam as a function of time be represented by \( H(t) \). Let the temperature of the oven be \( M(t) \). Newton's law of cooling says that at
any given instant the rate of change of temperature of the yam is proportional to the difference in temperature between the yam and the oven. As a differential equation this is
\[
\frac{dH}{dt} = \alpha(H(t) - M(t)).
\]
The yam should be warming if the oven is hotter than the yam is. That is to say, \(\frac{dH}{dt}\) should be positive when \(M > H\). This tells us that \(\alpha\) should be negative. It is traditional to use positive constants and explicit signs, so we set \(k = -\alpha\) and write
\[
\frac{dH}{dt} = -k(H(t) - M(t)), \quad k > 0.
\]

Even before solving Equation (25), we may see that the yam is in equilibrium (i.e. \(dH/dt = 0\)) exactly when it is the same temperature as the oven \((H = M)\).

If \(M(t) = M\) is constant, then Equation (25) is separable, and we solve it in the following sequence of formulas.

\[
\frac{dH}{H - M} = -k \, dt
\]
\[
\ln |H - M| = -kt + C
\]
\[
H - M = Be^{-kt}
\]
\[
H = M + Be^{-kt}
\]

(26)

Evaluating at \(t = 0\) we see that \(H(0) = M + B\), so \(B\) determines and is determined by the temperature of the yam at time 0. We see that the temperature rises or falls to the temperature of the oven via exponential decay. The graph of typical solutions for given \(k\) and \(M\) and variable \(H(0)\) are presented in Figure 8.

If \(M(t)\) is variable our model is one of an oven which was not preheated (so \(H(t)\) gradually increases to the set point) or one whose temperature is under constant adjustment. For a concrete example, imagine an object outside under the effect of a changing temperature as the sun rises and sets and the seasons change. In this case, we get a nonseparable equation, but one which is first order and linear. Thus we may apply the method of the previous section. Writing Equation (25) in standard form we get
\[
H'(t) + kH(t) = kM(t).
\]
Our integrating factor is then \( e^{kt} \) and the usual method yields the following calculations.

\[
\begin{align*}
\frac{d}{dt} \left( e^{kt} H(t) \right) &= ke^{kt} M(t) \\
\int ke^{kt} M(t) \, dt &= e^{kt} H(t) \\
H(t) &= e^{-kt} \int ke^{kt} M(t) \, dt
\end{align*}
\]

For general ambient temperature functions, \( M(t) \), this is as far as we can go; the integral is generally not computable in elementary terms.

One \( M(t) \) for which we can do the integral is given by a simple periodic ambient temperature: \( M(t) = A \sin(\omega t) \), where \( A \) and \( \omega \) are constants giving the amplitude and frequency of the oscillations of the ambient temperature around a base temperature of 0. (Without loss of generality, we may choose our temperature scale so that 0 is the median of the oscillations.) Using Corollary 8.15 (which we shall discuss later in the course), integration by parts, an integrating table, or your favorite reliable symbolic integrator,
we can continue our calculation as follows.

\[
H(t) = e^{-kt} \int A ke^{kt} \sin(\omega t) \, dt
\]

\[
= A ke^{-kt} \left[ \frac{e^{kt}}{k^2 + \omega^2} (k \sin(\omega t) - \omega \cos(\omega t)) + C \right]
\]

\[
(27) = \frac{Ak}{k^2 + \omega^2} (k \sin(\omega t) - \omega \cos(\omega t)) + CAke^{-kt}.
\]

We can deduce several interesting things by examining the form of this solution. When \( t = 0 \) we see \( C \) determines and is determined by the initial temperature, much as before. When \( \omega = 0 \) we achieve the solution, Equation (26), we had before with \( M = 0 \). As \( \omega \) increases in magnitude, a non-transient oscillatory response is apparent in the formula (27). This non-transient response is the same frequency as the driving temperature, but it is phase shifted. For large \( \omega \) the non-transient response approaches a phase shift of 90 degrees—that is to say \( H(t) \) oscillates like \( \cos(\omega t) \) when \( M(t) \) oscillates like \( \sin(\omega t) \) and \( \omega \) is large. We also see that the non-transient response becomes very small as \( \omega \) becomes very large; vibrating the knob on my oven very quickly has little effect on my yam. Figure 9 has a graph of solutions for given \( A \), \( k \) and \( \omega \), with \( H(0) \) varying through a few integral values. The graph of the oven temperature \( M(t) = A \sin(\omega t) \) is also shown. The reader is advised to reread our analysis of the behavior of the formula given at (27) and see each of our observations in the graphs of the solutions displayed in Figure 9.

4.1. Exercises.

Exercise 4.1. Notice in Figure 9 that \( H(t) \) is decreasing whenever its graph is above that of \( M(t) \). You should be able to see that this observation is reflected in equation (25). Write a few clear sentences explaining this.

Exercise 4.2. Suppose that the oven itself is cooling to a fixed temperature. We'll choose our temperature scale so that the ultimate temperature of the oven is 0. Then \( M(t) = Ae^{-\ell t} \). Solve Equation (25) for this \( H(t) \). Relate the constant that appears in the general solution to the initial temperature of the yam. Use the plotting program of your choosing to make a graph of the solutions for illustrative values of the constants.
Figure 9. Solutions to $H' = -k(H - A \sin(\omega t))$ for $k = 1$, $A = 1$ and $\omega = 2$.

5. NON-HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

We assume that the reader has become familiar with homogeneous second order linear differential equations—they are treated in the main text for the course. Here we briefly consider the non-homogeneous case. That is to say, we consider a differential equation of the form

$$\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = R(x).$$

What makes Equation (28) non-homogeneous is the presence of a non-zero term, $R(x)$. The equation

$$\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

is called the homogeneous equation associated to Equation (28). Note that if $y_0(x)$ and $y_1(x)$ are any two solutions to the non-homogeneous Equation (28), then $y_1 - y_2$ is a solution to the associated homogeneous Equation.
That is to say $y_1(x) = y_0(x) + f(x)$ where $f(x)$ is a solution to the associated homogeneous equation. This allows us to apply the following method to solve non-homogeneous second order linear differential equations with constant coefficients as presented in Equation (28):

1. Find the general solution, $y_{\text{hom}}(x)$, to the associated homogeneous Equation (29).
2. Find any one particular solution, $y_{\text{part}}(x)$, to Equation (28). Use guess and check. (There is a method called "variation of parameters" which finds such a particular solution, however we won’t discuss it. In any case, guess and check is quite often the easiest way to proceed.)
3. The general solution to the original Equation (28) is now $y(x) = y_{\text{part}} + y_{\text{hom}}$.
4. In an initial value or boundary value problem, use the initial or boundary conditions to solve for the constants in the general solution, just as we have already learned to do for first order linear equations and for homogeneous linear second order equations with constant coefficients.

**Example 5.1.** Here is a simple example to illustrate the method. We'll find the general solution to the differential equation

$$y'' + 2y' + y = x.$$ 

The characteristic polynomial is $\alpha^2 + 2\alpha + 1 = (\alpha + 1)^2$. Thus the exponential solution to the associated homogeneous equation is $e^{-x}$ and an independent solution is $xe^{-x}$. So the general solution to the associated homogeneous equation is $y = C_1e^{-x} + C_2xe^{-x}$. To find a particular solution to the original inhomogeneous equation we guess $y = ax + b$. This guess gives $y' = a$ and $y'' = 0$. Plugging into the original equation we get the equation

$$2a + ax + b = x$$

which is satisfied if $a = 1$ and $b = -2$. Thus a particular solution is $y = x - 2$ and the general solution is

$$y = C_1e^{-x} + C_2xe^{-x} + x - 2.$$ 

5.1. **Periodic driving force.** As a more extended example, we consider a model for a yam on a spring with a periodic driving force. Newton’s equation
for such a yam becomes

\[ m \frac{d^2 s}{dt^2} = -ks + A \sin(\eta t). \]

Here, \( m \) and \( k \) are positive constants denoting the mass of the yam and the spring constant of the spring. We have seen in an earlier lecture that \( \omega = \sqrt{\frac{k}{m}} \) is the frequency (in radians per time unit) at which the undriven yam will oscillate. \( A \) and \( \eta \) are constants giving the amplitude and frequency of the driving force. Putting our equation in the standard form we get

\[ \frac{d^2 s}{dt^2} + \frac{k}{m} s = \frac{A}{m} \sin(\eta t). \]

(30)

The general solution to the associated homogeneous equation is (as we have seen)

\[ s_{\text{hom}}(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t). \]

It is very reasonable to guess \( s = D \sin(\eta t) \) as a particular solution. Observation shows that driven yams respond at the driving frequency, and we can see that we will get three terms with \( \sin(\eta t) \) in them and hope to solve for \( D \) in the result. So we guess: We plug \( s = D \sin(\eta t) \) into Equation (30) and hope that we can solve for \( D \) to get a solution. The resulting equation is

\[ -D\eta^2 \sin(\eta t) + \omega^2 D \sin(\eta t) = \frac{A}{m} \sin(\eta t). \]

(31)

We look for a value of \( D \) which makes the left hand side of Equation (31) equal to the right hand side. As anticipated, there is a common factor \( \sin(\eta t) \). When we divide through by it and factor out the common \( D \) we get

\[ \left( \frac{1}{\omega^2 - \eta^2} \right) D = \frac{A}{m}. \]

(32)

If \( \omega \neq \pm \eta \) we can solve Equation (32) for \( D \). We get

\[ D = \frac{A/m}{\omega^2 - \eta^2}. \]

(33)

(We will treat the case \( \omega = \pm \eta \) later.)

Thus, when \( \omega \neq \pm \eta \), the general solution to Equation (30) is

\[ s(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{A/m}{\omega^2 - \eta^2} \sin(\eta t). \]

(34)

For a boundary value problem or an initial value problem we can solve for \( C_1 \) and \( C_2 \) in terms of the given data to get a unique solution. For example, in an initial value problem, \( s(0) = s_0 \) and \( s'(0) = v_0 \) are specified. We then can
take $t = 0$ in Equation (34) to see that $C_1 = s_0$. Differentiating Equation (34) now gives us

$$s'(t) = -s_0 \omega \sin(\omega t) + C_2 \omega \cos(\omega t) + \frac{\eta A/m}{\omega^2 - \eta^2} \cos(\eta t).$$

Taking $t = 0$ in Equation (35), we get

$$v_0 = C_2 \omega + \frac{\eta A/m}{\omega^2 - \eta^2}$$

which determines $C_2$ in terms of $v_0$. See the exercises for a simple concrete example. It is worth noticing that all of these solutions to Equation (30) have increasingly large amplitudes as the driving frequency $\eta$ approaches the resonant frequency $\omega$. This is reflected in the observed physical behavior of such yams.

5.2. Resonance. We now consider solutions of Equation (30) in the case where $\eta = \pm \omega$. In terms of the system we are modeling by this equation, we are now considering the case where the frequency of the driving force is equal to the resonant frequency. Physical experience of such systems is that they still respond at the driving frequency, but that they are unstable—the amplitude of the response grows larger and larger over time.

When $\omega = \pm \eta$, our analysis breaks down when we try to pass from Equation (32) to Equation (33). In fact, when $\omega = \pm \eta$, we see that the only way to satisfy Equation (32) is to have $A = 0$, which would mean there is no driving force at all. What we deduce from this is that in the presence of a driving force with $\omega = \pm \eta$, the guess $s = D \sin \eta t$ for the particular solution is no good—we need a different way to arrive at a particular solution.

One way of proceeding is to make a different guess, perhaps basing our guess on the observed behavior of resonantly driven yams. We ask you to carry this out in the exercises.

Here, we will show another way to arrive at the same solution. We will examine the solutions given at equation (34) and let $\eta \rightarrow \omega$. This method will also allow us to say some interesting things about the case when the driving frequency is close to (but not equal to) the resonant frequency.

Since the constants, $C_1$ and $C_2$, in Equation (34) depend on $s_0$ and $v_0$ as well as $\eta$ and $\omega$, we can simplify our work by setting particular values for $s_0$ and $v_0$ before continuing.
Factoring out what we can factor out, we can write the $s(t)$ given by Equation (34) when $s_0 = v_0 = 0$ as

$$s(t) = \frac{A/m}{\omega^2 - \eta^2} \left( \sin(\eta t) - \frac{\eta}{\omega} \sin(\omega t) \right).$$

(We ask you to verify this in the exercises.) Now, as we have admitted, this does not make sense when $\omega = \eta$, that is to say when the force driving the yam is the same as the frequency at which the unforced yam vibrates. Nevertheless, we can use the formula we have derived to examine what happens in the limit as $\eta \to \omega$. Clearly, the coefficient out front, $\frac{A/m}{\eta^2 - \omega^2}$, goes to infinity as $\eta$ approaches $\omega$. However, the term inside the parentheses also goes to zero in the limit. This is a job for L’Hospital’s rule! To apply the rule in this case we consider the expression for $s(t)$ as a function of $\eta$ with all other values constant and calculate

$$\lim_{\eta \to \omega} \frac{A \sin(\eta t) - \frac{\eta}{\omega} \sin(\omega t)}{\omega^2 - \eta^2} = \lim_{\eta \to \omega} \frac{A \frac{d}{d\eta}(\sin(\eta t) - \frac{\eta}{\omega} \sin(\omega t))}{\omega^2 - \eta^2}$$

$$= \lim_{\eta \to \omega} \frac{A \cos(\eta t) - \frac{1}{\omega} \sin(\omega t)}{-2\eta} = \frac{A \cos(\omega t) - \frac{1}{\omega} \sin(\omega t)}{-2\omega}.$$

There are (at least) two things we can do with this last expression. First of all, we can use the usual trick to express the sum of two sinusoidal functions of the same frequency as a single sinusoidal function with phase shift, $\phi = \arctan(-\omega t)$ and Pythagorean amplitude:

$$\frac{A \cos(\omega t) - \frac{1}{\omega} \sin(\omega t)}{-2\omega} = -\frac{A}{2\omega m} \sqrt{t^2 + \frac{1}{\omega^2}} \sin(\omega t + \phi).$$

This shows that our limiting $s(t)$ oscillates under the enveloping hyperbola $s = \frac{-A}{2\omega m} \sqrt{t^2 + \frac{1}{\omega^2}}$. On the other hand we can recognize that the expression we obtained,

$$\frac{A \cos(\omega t) - \frac{1}{\omega} \sin(\omega t)}{-2\omega}$$

is the sum of a solution, $\frac{A}{2\omega m} \sin(\omega t)$, to the homogeneous equation, $s'' + \omega^2 s = 0$, and another term,

$$-\frac{A}{2\omega m} \cos(\omega t).$$

If all of our limiting calculations work, the formula at (36) should be a particular solution to Equation (30) when $\eta = \omega = \sqrt{\frac{A}{m}}$. That is to say, it
is a solution of
(37) \[ s'' + \omega^2 s = \sin(\omega t). \]
The reader may check directly that \( \frac{A}{2\omega m} t \cos(\omega t) \) does indeed solve Equation (37).

Thus the general solution to Equation (37) is given by
(38) \[ s(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{-A}{2\omega m} t \cos(\omega t). \]

It is interesting to observe that every one of the solutions given in Equation (38) are unbounded as \( t \) grows bigger. Thus, regardless of initial conditions, when a yam is driven at its resonant frequency, its amplitude grows approximately linearly over time.

In Figure 10 we show the graphs of the resonant solution \( (\omega = \eta) \) together with a near resonant solution \( (\omega \approx \eta) \). The near resonant solution illustrates the phenomenon of “beats.” That is to say, one can see a periodic impulse with frequency equal to the difference \( \omega - \eta \) between the resonant frequency and the driving frequency. For more on this and further references, one may consult a text on differential equations. The standard introductory text, [BD65], is a good place to start.

5.3. Exercises.

Exercise 5.1. Find the general solution to
\[ y'' + y' + y = 17. \]

Exercise 5.2. More generally, find a particular solution to the equation
(39) \[ y'' + by' + cy = R \]
where \( b, c, \) and \( R \) are constants. (Your answer, of course, is likely to depend on the constants.) What does this tell you about the relationship between solutions to Equation (39) and solutions to the associated homogeneous equation? Interpret all of this in terms of the behavior of physical systems.

Exercise 5.3. Find the general solution to the equation
\[ y'' + y' - 6y = \cos x. \]

Exercise 5.4. Find the general solution to
\[ y'' - y = x^2. \]
\[ s'' + \omega^2 s = \sin(\omega t) \]

with \( y(0) = y'(0) = 0 \) together with a solution to \( s'' + \omega^2 s = \sin(\eta t) \)

where \( \eta \approx \omega \) satisfying the same initial conditions.

**Exercise 5.5.** Solve the initial value problem given by Equation (30) and the initial conditions \( s_0 = v_0 = 0 \).

**Exercise 5.6.** Guess \( s = Dt \cos \omega t \) as a particular solution to Equation (30) when \( \eta^2 = \frac{k}{m} \). Solve for \( D \). Relate your answer to our discussion above of the resonant yam.

### 6. A Description of the Complex Numbers

In this section we continue to extract from Apostol [Apo67]. However, [Apo67] goes into less detail than we would like to. I have used in addition the excellent books by Palka [Pal91] and Cartan [Car63]. [Pal91] is the best comprehensive book I know on complex analysis and [Car63] is the best short book. Both have much more material than is given in these notes. Another very good complex analysis text—the one I first learned from—is [Ahl78]. Whenever the reader would like to know more than what I have put here, I strongly recommend going to one of these sources.

#### 6.1. Real Numbers

In the typical calculus class, we studiously avoid defining the real numbers, \( \mathbb{R} \). Instead, we deal with \( \mathbb{R} \) using its "known"
properties. By “known,” I mean properties which are either both quite familiar and quite plausible, or—if unfamiliar—are at least quite plausible. This approach is both reasonable and desirable, because a careful definition of $\mathbb{R}$ brings us beyond the levels of rigor and abstraction to which we are accustomed. Using the properties, without proof, on the other hand feels natural and concrete. We now clarify this approach by enumerating the properties of $\mathbb{R}$ which we use. We will mention, but not prove that these properties completely characterize $\mathbb{R}$. The proof is in standard analysis texts, or courses, to which we refer the interested reader.

**Description 6.1.** $\mathbb{R}$ is a set equipped with:

1. Two special elements $0, 1 \in \mathbb{R}$.
2. Two binary operations $+$ and $\times$. ($\times$ is often written $\cdot$ or omitted altogether.)
3. An order relation, $<$. ($x < y$ is often written $y > x$.)

This equipment is endowed with numerous properties which describe the behavior of the real numbers. In the following the symbols $x, y,$ and $z$ denote elements of $\mathbb{R}$. Unless explicitly noted otherwise, the properties are being asserted to hold for any $x, y, z \in \mathbb{R}$. First we list the algebraic properties:

\[
\begin{align*}
(40) \quad & x + y = y + x \quad & xy = yx \\
(41) \quad & x + (y + z) = (x + y) + z \quad & x(yz) = (xy)z \\
(42) \quad & x + 0 = x \quad & x \cdot 1 = x \\
(43) \quad & 0 \neq 1 \\
(44) \quad & \forall x \exists y. x + y = 0 \quad & \forall x \neq 0 \exists y. xy = 1 \\
(45) \quad & x(y + z) = xy + xz
\end{align*}
\]

---

\[\text{In the list of properties, we have used the shorthand } \forall \text{ to stand for the phrase "for all," and the shorthand } \exists \text{ to stand for "there exists." The discussion below, in Example 6.7, should help to clarify this usage. Such abbreviations should be used with caution. They are standard in formal mathematical logic, and common as shorthand for the chalkboard or in one's own personal notes. Generally speaking these abbreviations are fine in displayed formula (which would be made too long by the insertion of the English phrase) but should be eschewed in written exposition.}\]
Next we list the properties relating to the ordering of the reals—including properties which show how the order relates to the algebra:

\[
\forall x, y \begin{cases} 
  x < y \\
  \text{or } x = y \\
  \text{or } y < x 
\end{cases}
\]

(46)

\[ x \not< x \]

(47)

\[ x < y, y < z \implies x < z \]

(48)

\[ x < y \implies x + z < y + z \]

(49)

\[ x < y, z > 0 \implies xz < yz \]

(50)

The final property is called the "least upper bound property." In words this property states that whenever \( S \subseteq \mathbb{R} \) has an upper bound (that is to say, there is some \( M \in \mathbb{R} \) which is bigger then every \( s \in S \)) the \( S \) has a least upper bound (that is to say, one which is smaller than every other one.) We can abbreviate this property in the following way:

\[ S \subseteq \mathbb{R} \text{ has an u.b.} \implies S \text{ has a l.u.b} \]

(51)

Thus the description 6.1 can be summarized as the real numbers are a set in which addition, multiplication, and ordering are defined which satisfy all of the properties listed from (40) to (51).

There is some standard mathematical vocabulary which allows us to describe \( \mathbb{R} \) in a fairly short sentence. We'll give a brief description of this terminology here.

**Definition 6.2.** A *field* is a set, \( F \) equipped with two special elements, \( 0_F \) and \( 1_F \) and two binary operations \( +_F \) and \( \times_F \) satisfying the conditions (40) through (45).

**Definition 6.3.** A *totally ordered set* is a set \( S \) with a relation \( < \) which satisfies properties (46) through (48).

**Definition 6.4.** An *ordered field* is a field which is also an ordered set and which satisfies the conditions at (50).

Thus Description 6.1 can thus be pithily summed up by saying that the real numbers are a totally ordered field satisfying the least upper bound property.
We will not prove or even carefully state the following theorem or its corollary. One may find proofs and proper statements in standard analysis texts. See for example [Rud76].

Vague Theorem 6.5. There is a set \( R \) with the properties listed in Description 6.1. There is only one.

Corollary 6.6. All properties of \( R \) may be derived from those listed in Description 6.1.

Example 6.7. We give a simple example to illustrate the flavor of the development of other well-known properties from the listed ones. We will show how the operation of subtraction may be obtained from the given structure. First of all, suppose that \( x + y_1 = x + y_2 = 0 \). Then the listed properties allow the following deduction that \( y_1 = y_2 \).

\[
y_2 = 0 + y_2 = (x + y_1) + y_2 = x + (y_1 + y_2) = x + (y_2 + y_1) = (x + y_2) + y_1 = 0 + y_1 = y_1
\]

Thus, we may improve on the property listed on the left side of (44): For every real number \( x \) there is exactly one real number \( y \) such that \( x + y = 0 \). We define \( -x \) to be this \( y \). We define \( x_1 - x_2 := x_1 + (-x_2) \).

Many other properties of fields follow from the axioms listed from (40) to (45). These and related issues are central topics in the study of abstract algebra—the interested student should consider our Math 67 and the courses which follow it. Similarly many other properties of the real numbers follow from the whole list of axioms (40) to (51). These and related issues are central topics in the study of real analysis—the interested student should consider our Math 63 and the courses which follow it.

We will leave all those issues to other courses; in this class, we will be happy to have a complete, moderately succinct, description of the real numbers. We will build an understanding of complex numbers on this description of the real numbers.

6.2. Exercises.

Exercise 6.1. Show how division can be obtained from the properties listed in Description 6.1.
6.3. The definition of the complex numbers. We will now take $\mathbb{R}$ as given. We will build $\mathbb{C}$ from $\mathbb{R}$. We give the details of this construction. (We'll still keep up our habit of omitting most proofs, leaving them for later courses.) Roughly speaking, $\mathbb{C}$ is obtained from $\mathbb{R}$ by adding in an element $i$ such that $i^2 = -1$. To give a careful definition, we proceed as follows.

Definition 6.8. $\mathbb{C}$ is the set defined by

$$C = \{a + bi \mid a, b \in \mathbb{R}\}.$$  

(At this point in the development, some authors write $(a, b)$ for $a + bi$ to emphasize that we have only defined $\mathbb{C}$ as a set so far; the $+$ sign does not (yet) represent addition, it is just symbolic fluff, like the comma in ordered pair notation.) $\mathbb{C}$ has two special elements $0_\mathbb{C}$ defined by $0_\mathbb{C} := 0 + 0i$ and $1_\mathbb{C}$ defined by $1_\mathbb{C} := 1 + 0i$. $\mathbb{C}$ has two binary operations, $+_\mathbb{C}$ and $\times_\mathbb{C}$ defined by

$$(a + bi) +_\mathbb{C} (c + di) := (a + c) + (b + d)i$$

$$(a + bi) \times_\mathbb{C} (c + di) := (ac - bd) + (ad + bc)i.$$  

$\mathbb{C}$ has a unary operation (called conjugation) defined and notated by

$$a + bi := a - bi.$$  

When we are told about $z \in \mathbb{C}$ but not told the $a, b \in \mathbb{R}$ for which $z = a + bi$, we will write $Re z$ and $Im z$ for such $a$ and $b$. In other words for every $z \in \mathbb{C}$ we have $Re z, Im z \in \mathbb{R}$ and $z = Re z + i \, Im z$.

When we think there is no risk of undue confusion, we will omit the subscripts on $0, 1, +,$ and $\times$, expecting the reader to know what we mean from context. We will make the same conventions regarding omitting the multiplication sign, precedence of evaluation and so on as are usual in the reals. These operations behave much “as expected.” In particular, we have the following Theorem.

Theorem 6.9. $\mathbb{C}$ together with $0, 1, +, \times$ is a field.

This is not a hard theorem to prove, but we omit its proof here. Instead we do some examples to make things clear. In these examples, and from now on we will make the convenient abbreviations $a$ for $a + 0i$ and $bi$ for $0 + bi$. In particular, this allows us to think of $\mathbb{R}$ as a subset of $\mathbb{C}$.

Here is a first example of multiplication:

$$(1 + 2i)(3 + 4i) = (3 - 8) + (6 + 4)i = -5 + 10i.$$
For a second example, let's look at the powers of $i$. We have

\[
\begin{align*}
  i^0 &= 1 \quad \text{(By convention.)} \\
  i^1 &= i \cdot 1 = i \\
  i^2 &= ii = -1 \quad \text{(As advertised.)} \\
  i^3 &= i(i^2) = i(-1) = -i \\
  i^4 &= i(i^3) = i(-i) = -i^2 = 1 \\
  i^5 &= i(i^4) = i \cdot 1 = i \\
  i^6 &= i(i^5) = ii = -1 \\
  \vdots
\end{align*}
\]

As another example, let's show how division may be accomplished. First of all, division by a real number $a = \alpha + \beta i$ may be accomplished by just dividing the parts: $(b + ci)/a = (b/\alpha) + (c/\alpha)i$. This together with the fact that for every complex number $z = a + bi$, we have

\[
z \overline{z} = (a + bi)(a - bi) = a^2 + b^2
\]

allows for the following formula for $z^{-1}$.

**Proposition 6.10.** For every $z = a + bi \neq 0$ in $\mathbb{C}$ we have

\[
z^{-1} = \frac{z}{z \overline{z}} = \frac{a - bi}{a^2 + b^2}.
\]

**Proof.**

\[
\frac{z}{z \overline{z}} = \frac{z \overline{z}}{z \overline{z}} = 1.
\]

Here are some examples of division.

\[
\begin{align*}
  (1 + i)^{-1} &= (1 - i)/2 = \frac{1}{2} - \frac{1}{2}i \\
  i^{-1} &= (0 + i)^{-1} = (0 - i)/(1) = -i \quad \text{(As noted already.)} \\
  (3 + 4i)^{-1} &= (3 - 4i)/25.
\end{align*}
\]

Here are some worked example calculations involving the arithmetic in $\mathbb{C}$. In each case, we rewrite the given expression in the form $a + bi$ where $a$ and $b$ are real.

a) $\left(1 + i\right)(2 + 3i) = 2 - 3 + 5i = -1 + 5i$. 
b) \[
\frac{1+i}{2+3i} = \frac{1+i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2+3-i}{13} = \frac{5-i}{13}.
\]

c) \((1+i)^{200} = ((1+i)^2)^{100} = (2i)^{100} = 2^{100}i^{100} = 2^{100}(i^4)^{25} = 2^{100}.
\]
d) \[
\sqrt{1+i} = \pm \left( \sqrt{\frac{1+i}{2}} + \frac{\sqrt{2-1}}{2} \right).
\]

The last calculation could be checked by squaring the right hand side. However the reader is probably interested in how we arrived at the expression on the right. Here is one way. (We give another way below, in Subsection 6.6, after we have observed a few more properties of the complex numbers.) We solve \((a+bi)^2 = (1+i)\) for \(a\) and \(b\). Performing the square and equating the real and imaginary parts gives the following two equations in the real unknowns \(a\) and \(b\).

\[
a^2 - b^2 = 1
\]
\[
2ab = 1
\]

There is no solution if \(a = 0\). If \(a \neq 0\), the second equation gives \(b = \frac{1}{2a}\) which we may plug into the first equation to get \(a^2 - \left(\frac{1}{2a}\right)^2 = 1\). Clearing the denominator and suggestive collection gives \((a^2)^2 - (a^2) - \frac{1}{4} = 0\). To this we can apply the quadratic formula to get \(a^2 = \frac{1 \pm \sqrt{1+1}}{2} = \frac{1 \pm \sqrt{2}}{2}\). (We ignore the negative solution, since \(a\) is to be real, so \(a^2\) must be positive.)

This gives \(a = \pm \sqrt{\frac{1+\sqrt{2}}{2}}\) and \(b = \frac{1}{2a} = \sqrt{\frac{\sqrt{2}-1}{2}}\). Thus our final answer is

\[
\sqrt{1+i} = \pm \left( \sqrt{\frac{1+\sqrt{2}}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}} \right).
\]

6.4. Exercises.

**Exercise 6.2.** Express each of the following in the form \(a + bi\) with \(a\) and \(b\) real.

a) \((1 + 2i) + (3 + 4i)\).

b) \((1 + 2i)(3 + 4i)\).

c) \(\frac{1+2i}{3+4i}\).

**Exercise 6.3.** Express each of the following in the form \(a + bi\).

i) \(\frac{1+i}{1-i}\).

ii) \(\frac{4+3i}{2+i}\).
\[ (\frac{-1}{2} + \frac{\sqrt{3}}{2}i)^3. \]

**Exercise 6.4.** Express \(\sqrt{5 + 12i}\) as \(a + bi\) by solving the equation \((a + bi)^2 = 5 + 12i\).

**Exercise 6.5.** Is there any positive power of \(1 + 2i\) which is equal to 1? Why or why not?

**Exercise 6.6.** Is there any positive power of \(\frac{3+4i}{3}\) which is equal to 1? Why or why not?

6.5. The geometry of the operations in \(\mathbb{C}\). The operations in \(\mathbb{C}\) have been defined algebraically, however we may interpret them geometrically by plotting \(a + bi\) at the point with \(x = a\) and \(y = b\). This will give us valuable intuition. The careful reader will note, however, that some geometric notions have not been given careful definitions in our treatment. Thus arguing from the geometry to the properties of \(\mathbb{C}\) should not be considered rigorous (in our treatment) no matter how valuable it is to building our intuition.

In discussing the geometry of the operations in \(\mathbb{C}\) it will be useful to have a “polar” description of the plane. We have drawn a picture of this representation in Figure 11.

**Definition 6.11.** For any complex number \(z = a + bi\) we define

\[ |z| := \sqrt{2z} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}. \]

In words, we call \(|z|\) the *modulus* of \(z\).

Thus \(|z|\) is the distance from 0 to \(z\).

**Vague Definition 6.12.** If \(z = a + bi\) is any non-zero complex number, we will write \(\arg z\) for the counter-clockwise angle between the line joining 0 to \(z\) and the \(x\)-axis. This angle, as is usual, is only defined up to a multiple of \(2\pi\). (So \(\theta, \theta + 2\pi, \theta - 2\pi \cdots\) all represent the same angle.) In words, we call \(\arg z\) the *argument* of \(z\).

In Figure 11, we have indicated \(\arg z\) by \(\theta\).

The reason we call our definition of argument “vague” is that it depends on the notion of angle which is geometric, and does not follow simply from our description of the real and complex numbers. We will indicate later (in Definition 8.22) how one might choose to give a rigorous definition of argument which depends only on the description of complex numbers as given. For now, we will rely on our geometric intuition.
Figure 11. In this picture we show $z = 2 + i$ together with its modulus and argument. The modulus, $|z|$, of $z$ is the distance from the origin to the point represented by $z$. The argument $\arg z$ is the angle—marked $\theta$ in the figure—between the segment joining the origin to $z$ and the x-axis.

6.5.1. The geometry of conjugation. By definition, if $z = x + yi$ the conjugate of $z$ is $\bar{z} = x - iy$. The geometry of conjugation is shown in Figure 12. In the figure we have shown $z$ and $\bar{z}$ when $z = 2 + i$. Observe that conjugation is reflection through the x-axis. Thus, the points in the dotted 'tjh' are all the results of conjugating the corresponding points in the solid 'tjh.'

6.5.2. The geometry of addition. By definition, if $z = x + yi$ and $w = a + bi$ we have $z + w = (a + x) + (b + y)i$. The geometry of this addition is shown in Figure 13. In this figure we have shown $z$, $w$, and $z + w$ when $w = 1 + 4i$ and $z = 2 + i$. We have also drawn directed segments or "vectors" from 0 to $z$, from 0 to $w$, from $z$ to $z + w$, and from $w$ to $z + w$. Observe that complex addition is "vector addition" in the sense that to obtain the position of $w + z$
Figure 12. Conjugation in the complex plane is reflection through the x-axis. In this picture we show the result of conjugating \( z = 2 + i \). The dotted 'tjh' is the result of conjugating each of the points in the solid 'tjh'.

we can place a translate of the vector from 0 to \( w \) so that its base rests on the tip of the vector from 0 to \( v \). Thus, the points in the dotted 'tjh' are all the results of adding the vector \( w \) to the corresponding points of the solid 'tjh'.

Stated verbally, addition of a constant complex number \( w = a + bi \) translates the plane by translation to the right by \( a \) units and up by \( b \) units.

6.5.3. The geometry of multiplication. By definition, if \( z = x + yi \) we have \( iz = -y + ix \). The geometry of this multiplication by \( i \) is shown in Figure 14. In this figure we have shown \( z \) and \( iz \) when \( z = 2 + i \). We have also drawn vectors from 0 to \( z \) and from 0 to \( iz \). Observe that multiplication by \( i \) is the geometric operation of rotation counterclockwise by one quarter of a full rotation. Thus, the points in the dotted 'tjh' are all the results of multiplying each of the points in the solid 'tjh' by \( i \).
FIGURE 13. Addition in the complex plane is “vector addition.” In this picture, we show the result of adding \( w = 1 + 4i \) to \( z = 2 + i \). The dotted ‘tjh’ is the result of adding \( w \) to each of the points in the solid ‘tjh.’

By definition, if \( z = x + iy \) and \( w = a + bi \) we have \( wz = (ax - by) + i(ay + bx) \). However, to understand the geometry, it is helpful to only multiply out part way: \( wz = w(x + iy) = xw + iyw \). Thus the point, \( z \), which is \( x \) units to the right of the origin and \( y \) units above the origin is transformed under multiplication by \( w \) to the point, \( wz \), which is \( x|w| \) in the direction of \( w \) from the origin and \( y|w| \) units in the direction of \( iw \) from the origin. The result is shown in Figure 15. We observe that multiplication by a complex constant \( w = a + bi \) rotates and expands the plane in such a way that \( 1 \) is transformed to \( w \) and \( i \) is transformed to \( iw \). The result is a rotated magnification or demagnification which may change size, but does not change angles or relative sizes.

As long as we are willing to use our non-rigorous notion of angle, we can also read the following off of our picture of complex multiplication.
Vague Theorem 6.13. If \( w, z \) are any two nonzero complex numbers, then

\[
\arg(wz) = \arg w + \arg z.
\]

In Figure 15, I have indicated \( \arg z \) by \( \theta \) and \( \arg w \) by \( \phi \). We will make this argument rigorous later in Corollary 8.23.

6.5.4. The geometry of reciprocals. Proposition 6.10 shows us that argument, \( \arg z^{-1} = \arg \frac{1}{z} = -\arg z \), and the modulus, \( \frac{|z|}{|z^*|} = |z|^{-1} \). We will call the transformation taking each \( z \in \mathbb{C} \) to \( 1/z \) " reciprocation." The geometry of reciprocation is indicated in Figure 16. In the figure we have shown \( z \) and \( 1/z \) when \( z = \frac{1}{4} + \frac{1}{2}i \). The points of the dotted 'tjh' are all the results of reciprocating the corresponding points in the solid 'tjh.' This geometry is very interesting. We can see that as Proposition 6.10 predicts, points outside the unit circle—that is to say, points with modulus greater than
Figure 15. Multiplication by $w = a + bi$ in the complex plane combines a rotation and a magnification or demagnification. In this picture, we show the result of multiplying $z = \frac{2}{3} + \frac{11}{18}i$ by $w = 3 + 2i$. The dotted 'tjh' is the result of multiplying each of the points in the solid 'tjh' by $w$. (Note that the scales in the two halves of the pictures are not the same.) We've also indicated the arguments of $w$, $z$, and $wz$.

1—are transformed to points inside the circle—that is to say, points with modulus less than 1. We can also see that points on a given line through the origin, are transformed to points on the reflection of that line through the $x$-axis. We can observe another things in the figure, which we will explain after we develop some more tools for dealing with complex numbers: Circles and lines (or segments of circles and lines) are transformed to circles and lines (or segments of circles and lines). Thus, the parallel top bar and cross bar of the solid 'tjh' are transformed into the perhaps concentric arcs of the corresponding parts of the dotted 'tjh.' Similarly, the circular hook on the solid 'tjh' is transformed to the circular hook on the dotted 'tjh.'

6.6. More on Conjugation and Modulus.

Proposition 6.14. For every $w, z \in \mathbb{C}$ we have $\bar{w} + \bar{z} = \bar{w + z}$ and $\bar{wz} = \bar{w} \bar{z}$.

Proof. The statement about addition is a simple consequence of the definitions and is left to the reader. The other statement is not much harder, but
Reciprocating (that is to say the transformation \( z \mapsto \frac{1}{z} \)) has very interesting geometry. In this picture, we show the result of reciprocating \( z = \frac{1}{3} + \frac{7}{6}i \). The dotted ‘tjh’ is the result of reciprocating each of the corresponding points in the solid ‘tjh.’

we spell it out. Let \( w = a + bi \) and \( z = x + yi \). We then have

\[
\overline{wz} = (a + bi)(x + yi) = (ax - by) + (bx + ay)i = (ax - by) - (bx + ay)i
\]

\[
\overline{wz} = (a + bi)(x + yi) = (a - bi)(x - yi) = (ax - by) - (bx + ay)i.
\]

\[\square\]

**Corollary 6.15.** \( |wz| = |w| \ |z| \).

**Proof.**

\[
|wz| = \sqrt{\text{wz} \overline{wz}} = \sqrt{\text{w} \overline{w} \sqrt{\text{z} \overline{z}}} = |w| \ |z|.
\]

\[\square\]
Combining Corollary 6.15 with the Vague Theorem 6.13 we get the following slogan: To multiply \( wz \) we “multiply the moduli and add the arguments.” (My first real math teacher told me that he was taught to chant in unison with his classmates to raise group morale.)

**Corollary 6.16.** \(|w^{-1}| = |w|^{-1}\).

**Proof.**

\[
|w| |w^{-1}| = |ww^{-1}| = |1| = 1.
\]

\(\square\)

**Example 6.17.** As an example of using these basic properties of conjugation and modulus, we show how to find \( \sqrt{a + bi} \) in terms of the real numbers \( a \) and \( b \). The result will be both simpler and more general than the derivation of \( \sqrt{1 + i} \) which we gave in the example at the end of Subsection 6.3. Suppose that \( z = \sqrt{a + bi} \), that is to say \( z^2 = a + bi \). We then have

\[
(\text{Re } z)^2 = \left( \frac{z + \overline{z}}{2} \right)^2 = \frac{z^2 + 2z\overline{z} + \overline{z}^2}{4} = \frac{z^2 + z\overline{z} + 2|z|^2}{4} = \frac{2\text{Re } (z^2) + 2|z|^2}{4} = \frac{a + \sqrt{a^2 + b^2}}{2}.
\]

It follows that

\[
\text{Re } z = \text{Re } \sqrt{a + bi} = \pm \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.
\]

Notice that no matter what real numbers \( a \) and \( b \), are used in equation (52) the square roots are all of non-negative real numbers. Similar reasoning gives a formula for the imaginary part of \( \sqrt{a + bi} \).

6.7. Exercises.

**Exercise 6.7.** Compute the modulus and argument of the numbers given in problems 1—8 of Appendix B in [HHGM +05]. (In fact, their \( r \) is exactly modulus and their \( \theta \) is exactly argument. We are going very soon to give a more careful treatment of complex exponentials, so we are being a bit more careful in our definitions of modulus and argument.)

**Exercise 6.8.** Find the modulus and argument of \( 1 + i \).

**Exercise 6.9.** Find the modulus and argument of \( \sqrt{1 + i} \).
Exercise 6.10. Prove that for every \( w, z \in \mathbb{C} \) we have the equality
\[
\overline{z + w} = \overline{z} + \overline{w}.
\]

Exercise 6.11. Prove that for every \( w, z \in \mathbb{C} \) we have the equality
\[
|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2.
\]
Interpret this result geometrically.

Exercise 6.12. Use an argument involving the properties of the modulus to show that there is no positive power of \( 1 + 2i \) which is equal to 1. Is there any positive power of \( \frac{1 + 4i}{3} \) which is equal to 1? Why or why not? (Why doesn't the modulus argument work? This problem is harder, but try to give a more precise argument than just the observation that the imaginary part never seems to be zero—give a convincing argument that that is indeed the case.)

Exercise 6.13. Complete the discussion begun in Example 6.17. In particular, use the same sort of reasoning to get a formula for \( \text{Im} \sqrt{a + bi} \), up to sign. There appear to be four possibilities altogether for how to choose the signs in the resulting formula for \( \sqrt{a + bi} \). Show that only two of the choices actually give square roots. Show that the formulas obtained reproduce the expression for \( \sqrt{1 + i} \) we gave in the example at the end of Subsection 6.3.

6.8. More on the geometry of reciprocation. In this subsection we explain why reciprocation takes lines and circles to lines and circles.

Lemma 6.18. The lines and circles in \( \mathbb{C} \) are exactly the solutions to equations of the form
\[
A z \bar{z} + B z + \bar{B} \bar{z} + C = 0
\]
where \( A, C \in \mathbb{R} \) and \( B \bar{B} > AC \).

Proof. Write \( z = x + yi \) and \( B = a + bi \) and expand Equation (53) to get
\[
A(x^2 + y^2) + 2(\bar{a}x - \bar{b}y) + C = 0.
\]
If \( A = 0 \), Equation (54) is the equation of a line. (If \( a \) and \( b \) are nonzero, this line is the one with \( x \)-intercept \( -\frac{1}{2a} \) and \( y \)-intercept \( \frac{C}{2b} \)). Otherwise it is parallel to one or the other axis.) If \( A \neq 0 \) we can complete the square: Multiply Equation (54) by \(-1\) if necessary and relabel so that \( A > 0 \). Then dividing by \( A \) and completing the square gives
\[
x^2 + 2x \frac{a}{A} + \left( \frac{a}{A} \right)^2 + y^2 - 2y \frac{b}{A} + \left( \frac{b}{A} \right)^2 = \left( \frac{a}{A} \right)^2 + \left( \frac{b}{A} \right)^2 - \frac{C}{A}.
\]
which when simplified becomes
\[
\left( x + \frac{a}{A} \right)^2 + \left( y - \frac{b}{A} \right)^2 = \frac{a^2 + b^2 - CA}{A^2}.
\]
This last equation is a circle as long as \( a^2 + b^2 = BB > AC \). Furthermore, we can get any circle this way, by just taking \( A = 1 \), choosing \( B \) to achieve the desired center of the circle, and choosing \( C \) to achieve the desired radius. □

Corollary 6.19. If \( C = \{ z \mid Az \bar{z} + Bz + \bar{B} \bar{z} + C = 0 \} \) is any circle or line in \( C \), then the set \( C' = \{ w = z^{-1} \mid z \in C \} \) is also a circle or a line. That is to say, reciprocation takes lines and circles to lines and circles.

Proof. Setting \( z = \frac{1}{w} \) in \( Az \bar{z} + Bz + \bar{B} \bar{z} + C = 0 \) and then multiplying through by \( w \bar{w} \) gives

\[
Aw^{-1} \bar{w}^{-1} + Bw^{-1} + \bar{B} \bar{w}^{-1} + C = 0
\]
\[
A + B \bar{w} + \bar{B} w + Cw \bar{w} = 0
\]
\[
C \bar{w} w + \bar{B} w + B \bar{w} + A = 0.
\]
However this is again the form of a circle. That is to say, it is Equation (53) with \( A \) and \( C \) interchanged, \( B \) replaced by \( \bar{B} \), and \( z \) replaced by \( w \). □

6.9. Exercises.

Exercise 6.14. In this problem, consider the following figure as a subset of the complex plane.

Draw axes and then indicate the result of performing the indicated operation to every point of the figure. Where it helps to clarify your drawings, give a brief explanation.
a) Replacing each \( z = a + bi \) in the figure with \( z + (1 + i) \).
b) Replacing each \( z = a + bi \) in the figure with \( (1 + i)z \).
c) Replacing each \( z = a + bi \) in the figure with \( z/(1 + i) \).
d) Replacing each \( z \) in the figure of Problem 3.2 with \( 1/z \). (You should assume that the figure is made out of segments of circles and lines.)

7. SEQUENCES AND SERIES REVISITED

In this section, we show how the facts about sequences and series treated earlier in the course extend to sequences and series of complex numbers. We will see that the convergence tests we already know for real series will often suffice in the extended complex context.

7.1. Series of complex numbers. Absolute convergence. In order to relate absolute convergence of complex series to absolute convergence of real series, we need the following simple lemma.

Lemma 7.1. For any \( z \in \mathbb{C} \) we have \( \Re z \leq |z| \) and \( |\Im z| \leq |z| \).

Proof. \( |z|^2 = |\Re z|^2 + |\Im z|^2 \). \( \square \)

Definition 7.2. If we are given complex numbers \( z_1, z_2, \ldots \in \mathbb{C} \) we define

\[
\sum_{k=1}^{\infty} z_k := \sum_{k=1}^{\infty} \Re z_k + i \sum_{k=1}^{\infty} \Im z_k.
\]

In particular, we say the left hand series converges exactly when both of the right hand series converge.

Theorem 7.3. If \( \sum_{k=1}^{\infty} |z_k| \) converges then so does \( \sum_{k=1}^{\infty} z_k \). In this case, we say \( \sum_{k=1}^{\infty} z_k \) converges absolutely.

Proof. This follows directly from the comparison test, Lemma 7.1, and Definition 7.2. In particular, the series of real and imaginary parts of the \( z_k \)'s are both absolutely convergent series of real numbers. \( \square \)

7.2. Complex power series. Now suppose \( a_0, a_1, \ldots \in \mathbb{C} \). We consider the series

\[
a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k=0}^{\infty} a_k z^k
\]
as a function of \( z \).

Lemma 7.4. Suppose \( r \in \mathbb{R} \) is such that \( \sum_{k=1}^{\infty} |a_k| r^k \) converges and that \( |z| \leq r \). Then \( \sum_{k=1}^{\infty} a_k z^k \) converges absolutely.
Proof. Apply the comparison test in the form $|a_k z^k| \leq |a_k|r^k$. \hfill $\Box$

Lemma 7.4 is all we will need in these notes. However, there is a fact which gives a little more precise information and which is not hard to prove. See any complex analysis book for a proof and other details.

Fact 7.5. Let $R$ be the radius of convergence of the real power series $\sum_{k=1}^{\infty} |a_k| r^k$. (This is a power series in the variable $r$. It is sometimes called the “associated real series.”) We then have

- When $|z| < R$, $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely.
- When $|z| > R$, $\sum_{k=0}^{\infty} a_k z^k$ diverges.
- When $|z| = R$, $\sum_{k=0}^{\infty} a_k z^k$ may converge for some points and diverge for others.

The set $\{z \in \mathbb{C} \mid |z| < R\}$ is called the disk of absolute convergence.

Example 7.6. The complex geometric series is

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + \ldots.$$ 

The associated real series is, in this case, $1 + r + r^2 + \ldots$ which has radius of convergence 1. Thus $1 + z + z^2 + \ldots$ converges absolutely for $|z| < 1$. In fact, since

$$(1 - z)(1 + z + \ldots + z^n) = 1 - z^{n+1},$$

we have

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$ 

So for $|z| < 1$ we have $\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$ just like the similar formula for the real geometric series.

Example 7.7. The complex exponential series is

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots.$$ 

The associated real series is, in this case, $1 + r + \frac{r^2}{2} + \frac{r^3}{6} + \ldots$ which converges for all $r \in \mathbb{R}$. Thus $1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots$ converges absolutely for all $z \in \mathbb{C}$. We will have much more to say about this series later in these notes.

We will often find ourselves considering complex power series with real coefficients. Here is one quick lemma about such series which we will soon find useful.
Lemma 7.8. If $a_0, a_1, \ldots \in \mathbb{R}$ and if $f(z) = a_0 + a_1z + a_2z^2 + \ldots$, then we have $f(z) = f(\bar{z})$ for every $z$ in the domain of convergence of $f$.

Proof. We have

$$f(z) = a_0 + a_1z + a_2z^2 + \ldots = a_{\bar{0}} + \bar{a}_1\bar{z} + a_2\bar{z}^2 + \ldots$$

$$= \bar{a}_0 + \bar{a}_1\bar{z} + \bar{a}_2\bar{z}^2 + \ldots = a_0 + a_1\bar{z} + a_2\bar{z}^2 + \ldots = f(\bar{z})$$

$\square$

7.3. Exercises.

Exercise 7.1. What is the disk of absolute convergence of the series $\sum_{n=0}^{\infty} z^n$?

8. Complex Exponential and Trigonometric Functions

8.1. The definition of the exponential and trigonometric functions.

In this section, we will use the properties of complex numbers and of series of complex numbers which we have derived above to give definitions of $e^z$, for every complex number $z$. It should be clear to the reader that for a general complex number $e^z$ cannot be defined as merely repeated multiplication of some number $e$ by itself $z$-times. On the other hand, we will give a simple power series definition of $e^z$ which holds for all complex numbers, and then show that $e^z$ behaves as desired when $z$ is an integer. As an interesting byproduct, we will also obtain rigorous definitions of angle, of the trigonometric functions, and of the real number $\pi$.

Thus, we ask the reader to willingly temporarily suppress any previous knowledge she may have of the exponential and trigonometric functions. By the end of these notes, we will have rebuilt these notions in the complex setting and also have shown that when restricted to the more familiar setting they recover the previously known concepts.$^9$

Definition 8.1.

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$^9$We hope that the resulting story will be entertaining in addition to exposing the reader to some new ideas and some new perspectives on old ideas. We provide proofs of many of our assertions in order to make our story complete. In this course, we will not be asking students to be able to create such proofs on their own, but rather to develop the skills necessary to read and understand the development as it is presented in lecture and in these notes.
In the face of typographical exigency, we often write \( \exp(z) \) for \( e^z \). When \( \theta \in \mathbb{R} \), we define real numbers \( \cos \theta \) and \( \sin \theta \) by the formula
\[
e^{i\theta} = \cos \theta + i \sin \theta.
\]
Equivalently,
\[
\cos \theta := \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta := \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]
We define the other trigonometric functions in terms of \( \sin \) and \( \cos \) as usual.
\[
\tan \theta := \frac{\sin \theta}{\cos \theta}, \quad \sec \theta := \frac{1}{\cos \theta}, \quad \cot \theta := \frac{\cos \theta}{\sin \theta}, \quad \csc \theta := \frac{1}{\sin \theta}.
\]
It is immediate that \( e^0 = \cos 0 = 1 \) and that \( \sin 0 = 0 \).

One may think that we had already defined \( \sin \) and \( \cos \) and used the definition to derive their power series in the first part of this course, and one would be partly right. We did derive Taylor series for \( \sin \) and \( \cos \) based on properties of \( \sin \) and \( \cos \) which we took for granted at that time. In these notes, however, we are now sketching a construction of the real and complex numbers from the ground up. Thus we make the definition above and we will be obliged to see that \( \sin \) and \( \cos \) behave as expected. The first thing to notice is that the power series we derived in the first part of the course are essentially equivalent to our current definition of \( \sin \theta \) and \( \cos \theta \).

**Corollary 8.2.** For each real number \( \theta \) we have
\[
\cos \theta = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \ldots
\]
\[
\sin \theta = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{5!} - \ldots.
\]

**Proof.** Using the Definition 8.1 we have
\[
\cos \theta + i \sin \theta = e^{i\theta}
\]
\[
= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots
\]
\[
= 1 + (i\theta) - \frac{(\theta)^2}{2} \quad i \frac{(\theta)^3}{6} + \frac{(\theta)^4}{4!} + \ldots
\]
\[
= 1 - \frac{(\theta)^2}{2} + \frac{(\theta)^4}{4!} + \ldots + i \left( \theta - \frac{(\theta)^3}{6} + \frac{(\theta)^5}{5!} + \ldots \right).
\]
The same argument may be expressed in summation notation as follows.\textsuperscript{10}

\[
\cos \theta + i \sin \theta = e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} i^k (\theta)^{2k} \frac{1}{(2k)!} + i \sum_{k=0}^{\infty} \frac{i^{2k+1} (\theta)^{2k+1}}{(2k+1)!}
\]

\[
= \sum_{k=0}^{\infty} (\theta)^{2k} \frac{(-1)^k}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k (\theta)^{2k+1}}{(2k+1)!}
\]

As a direct consequence of Lemma 7.8 we have the following.

Corollary 8.3. $e^z = e^z$.

8.2. Applications of $e^z e^w = e^{z+w}$. Soon we will prove the following theorem.

Theorem 8.4. For every $w, z \in \mathbb{C}$ we have

\[
e^z e^w = e^{z+w}.
\]

The proof of Theorem 8.4, while quite pretty, requires a lot of subscripts. So first we give some examples to show how useful the formula is. The slogan is that everything you ever wanted to know about trigonometry follows from a few basic facts about the exponential function. In particular, the proofs of these trigonometric facts depend only on Corollary 8.3 and Theorem 8.4. Later, after proving Theorem 8.4, we will give several more applications.

Corollary 8.5. For every $\theta \in \mathbb{R}$ we have $\sin^2 \theta + \cos^2 \theta = 1$.

\textsuperscript{10}Of course, we need only give the argument once. The student may choose for herself which presentation seems clearest in each instance. Sometimes summation notation will be clearest and sometimes the elliptical formulation ("..." is called an ellipsis.) is best.
\[ \sin^2 \theta + \cos^2 \theta = \left| e^{i\theta} \right|^2 = e^{i\theta} \overline{e^{i\theta}} = e^{i\theta} e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1. \]

\[ \boxed{\square} \]

**Corollary 8.6.** For every \( \theta, \phi \in \mathbb{R} \) we have

\[
\begin{align*}
\cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\
\sin(\theta + \phi) &= \sin \theta \cos \phi + \cos \theta \sin \phi.
\end{align*}
\]

**Proof.**

\[
\begin{align*}
\cos(\theta + \phi) + i\sin(\theta + \phi) &= e^{i(\theta + \phi)} = e^{i\theta} e^{i\phi} = (\cos \theta + i\sin \theta)(\cos \phi + i\sin \phi) \\
&= (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi).
\end{align*}
\]

\[ \boxed{\square} \]

8.3. **Exercises.** In the following exercises, feel free to use the “known” facts about \( \sin \) and \( \cos \), including their periodicity and values at standard rational multiples of \( \pi \). We have not yet proved all those facts from our definition. (We’ll sketch these proofs in a later section.)

**Exercise 8.1.** Express each of the following complex numbers in the form \( a + bi \).

a) \( e^{\pi i/2} \).

b) \( 3e^{\pi i} \).

c) \( e^{\pi i/4} - e^{-\pi i/4} \).

**Exercise 8.2.** Use \( e^{z+w} = e^z e^w \) to prove that \( e^z \neq 0 \) for all \( z \).

**Exercise 8.3.** Find all those complex numbers \( z \) for which \( e^z = 1 \).

**Exercise 8.4.** Use complex exponentials to derive a formula for \( \cos 3\theta \) in terms of \( \sin \theta \) and \( \cos \theta \).

8.4. Two “lemmas” in preparation of the proof of \( e^z e^w = e^{z+w} \). We need two lemmas, but the second one is usually called the binomial theorem; thus the quotation marks in the title of this subsection are required.
8.4.1. Products of Absolutely convergent series. First we state a general lemma about absolutely convergent series of complex numbers. We will not give the proof, but it can be found in the references. The following diagram will help to see what is going on.

<table>
<thead>
<tr>
<th></th>
<th>$b_0$ + $b_1$ + $b_2$ + $b_3$ + $b_4$ + ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$a_0b_0$ + $a_0b_1$ + $a_0b_2$ + $a_0b_3$ + $a_0b_4$ + ...</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$a_1b_0$ + $a_1b_1$ + $a_1b_2$ + $a_1b_3$ + $a_1b_4$ + ...</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_2b_0$ + $a_2b_1$ + $a_2b_2$ + $a_2b_3$ + $a_2b_4$ + ...</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3b_0$ + $a_3b_1$ + $a_3b_2$ + $a_3b_3$ + $a_3b_4$ + ...</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_4b_0$ + $a_4b_1$ + $a_4b_2$ + $a_4b_3$ + $a_4b_4$ + ...</td>
</tr>
<tr>
<td>...</td>
<td>... + ... + ... + ... + ... + ... + ...</td>
</tr>
</tbody>
</table>

Lemma 8.7. Suppose that $a_0 + a_1 + a_2 + \cdots = \sum_{p=1}^{\infty} a_p$ and $b_0 + b_1 + b_2 + \cdots = \sum_{q=1}^{\infty} b_q$ are absolutely convergent series of complex numbers. Define $c_k$ by the following equations.

\[
\begin{align*}
c_0 &= a_0b_0 \\
c_1 &= a_0b_1 + a_1b_0 \\
c_2 &= a_0b_2 + a_1b_1 + a_2b_0 \\
& \vdots \\
c_k &= a_0b_k + a_1b_{k-1} + \cdots + a_kb_0 = \sum_{p+q=k} a_pb_q.
\end{align*}
\]

Then $\sum_{k=0}^{\infty} c_k$ converges absolutely and

\[
\sum_{k=0}^{\infty} c_k = \left( \sum_{p=0}^{\infty} a_p \right) \left( \sum_{q=0}^{\infty} b_q \right).
\]

We won’t prove this, but we point out that it is eminently reasonable, since the terms in the $c_k$’s are exactly the products of pairs, one chosen from the $a_p$ and one chosen from the $b_q$ as you can see by looking at the lines of slope one in the diagram. (One does need to be a little careful though—the lemma is not generally true for series which are not absolutely convergent.)
convergent. Thus the proof is a little more delicate than one might hope. See any decent analysis book or class for details.) Finally, note that we can pithily sum up the lemma by the equation

\[ \left( \sum_{p=0}^{\infty} a_p \right) \left( \sum_{q=0}^{\infty} b_q \right) = \sum_{k=0}^{\infty} \left( \sum_{p+q=k} a_p b_q \right) \]

which holds whenever the series on the left hand side are absolutely convergent.

8.4.2. The binomial theorem. The other lemma we need is the binomial theorem. This is neither stated or proved in its classical form in our text, [HHGM+05]. We give a careful statement and proof here both in the interest of completeness, and to give an example of an inductive proof. (These details are not always included in the class lectures.) Recall that by conventional definition, \( x^0 = y^0 = 0! = 1 \).

Theorem 8.8. Suppose \( x, y \) are elements of a set equipped with operations \( + \) and \( \times \) satisfying the conditions listed above as Equations (40), (41), (42), and (45). (Such sets are called commutative rings.) Suppose also that \( n \) is a nonnegative integer, we then have

\[
(x + y)^n = \sum_{p+q=n} \frac{n!}{p!q!} x^p y^q,
\]

(In the last sum, the notation is meant to mean that the sum is over all pairs \( p, q \) of nonnegative integers for which \( p + q = n \).)

Proof. First notice that the right hand sides of the three equalities are all equal. The last one is just the second one in summation notation. The first one is equal to the second by comparison of the coefficients of the general
terms:

\[
P \text{ terms } \frac{n(n-1) \ldots (n-p+1)}{p!} \]

\[
P \text{ terms } \frac{n(n-1) \ldots (n-p+1)}{p!} \frac{(n-p)(n-p-1)(n-p-2) \ldots 3 \cdot 2 \cdot 1}{(n-p)(n-p-1)(n-p-2) \ldots 3 \cdot 2 \cdot 1} = \frac{n!}{p! \cdot q!}
\]

where we have used the fact that \( p + q = n \). We think of the assertion we are proving as a separate statement for each value of \( n = 0, 1, 2, \ldots \). For \( n = 0 \), the only way for \( p + q \) to equal \( n \) is for \( p \) and \( q \) to both be zero. So the zeroth instance of the assertion is \((x + y)^0 = 1\), which is certainly true. Although strictly not necessary for the proof, we check the first and second instances of the assertion as well. This will help make sense of the notation we are using. When \( n = 1 \) we can have two terms, \(|p, q| = (0, 1)\) and \(|p, q| = (1, 0)\). The resulting assertion is that

\[
(x + y)^1 = \frac{1!}{0!1!}x^0y^1 + \frac{1!}{1!0!}x^1y^0.
\]

That is to say, when \( n = 1 \), the assertion is that \( x^1 y + x^0 y^1 \), which is also certainly true. When \( n = 2 \) we get the usual expansion of \((x + y)^2\). (The reader who has not ever done this before should check this for herself.) Now what we will do is show that if the \( n \)-th instance of the assertion is true, so is the \( n + 1 \)-st. Since we have already checked the zeroth instance, this will allow us to deduce all the other instances, with the zeroth implying the first, the first implying the second, the second implying the third, and so on. The \( n \)-th instance of our assertion is

\[
(x + y)^n = \sum_{p+q=n} \frac{n!}{p!q!}x^py^q.
\]

From this we may deduce the \( n + 1 \)-st case through the following sequence of equalities. Throughout, keep in mind that \( p, q, p', q' \) range through all non-negative integer values for which the other conditions hold, except when
explicitly stated otherwise.

\[(x + y)^{n+1} = (x + y)(x + y)^n\]
\[= (x + y) \sum_{p+q=n} \frac{n!}{p!q!} x^p y^q\]
\[= x \sum_{p+q=n} \frac{n!}{p!q!} x^p y^q + y \sum_{p+q=n} \frac{n!}{p!q!} x^p y^q\]
\[= \sum_{p+q=n} \frac{n!}{p!q!} x^p y^q + \sum_{p+q=n} \frac{n!}{p!q!} x^p y^{q+1}\]
\[= \sum_{p'+q'=n+1, p' \neq 0} \frac{n!}{(p'-1)!q'} x^{p'} y^q + \sum_{p+q'=n+1, q' \neq 0} \frac{n!}{p!(q'-1)!} x^p y^{q'}\]
\[= x^{n+1} y^0 + \left( \sum_{p+q=n+1, p, q \neq 0} \left( \frac{n!}{(p-1)!q!} + \frac{n!}{p!(q-1)!} \right) x^p y^q \right) + x^0 y^{n+1}\]
\[= x^{n+1} y^0 + \left( \sum_{p+q=n+1, p, q \neq 0} \left( \frac{pn!}{p!q!} + \frac{q(n)!}{p!q!} \right) x^p y^q \right) + x^0 y^{n+1}\]
\[= \left( \sum_{p+q=n+1} \frac{(n+1)!}{p!q!} x^p y^q \right)\]

8.5. Proof of \(e^z e^w = e^{z+w}\).

Proof of Theorem 8.4. The theorem follows directly from the two lemmas. We give the details anyway. In Lemma 8.7 we take \(a_p = \frac{1}{p!} z^p\) and \(b_q = \frac{1}{q!} w^q\).
The illustrative table before Lemma 8.7 then becomes the following table.

<table>
<thead>
<tr>
<th>(1)</th>
<th>(1 + w + \frac{1}{2}w^2 + \frac{1}{3}w^3 + \cdots + \frac{1}{q!}w^q + \cdots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(1 + w + \frac{1}{2}w^2 + \frac{1}{3}w^3 + \cdots + \frac{1}{q!}w^q + \cdots)</td>
</tr>
<tr>
<td>(z)</td>
<td>(z + zw + z\frac{1}{2}w + \frac{1}{3}w^3 + \cdots + \frac{1}{q!}w^q + \cdots)</td>
</tr>
<tr>
<td>(\frac{1}{2}z^2)</td>
<td>(\frac{1}{2}z^2 + \frac{1}{2}z^2w + \frac{1}{2}z^2w^2 + \frac{1}{3}z^2w^3 + \cdots + \frac{1}{q!}z^2w^q + \cdots)</td>
</tr>
<tr>
<td>(\frac{1}{3}z^3)</td>
<td>(\frac{1}{3}z^3 + \frac{1}{3}z^3w + \frac{1}{3}z^3w^2 + \frac{1}{3}z^3w^3 + \cdots + \frac{1}{q!}z^3w^q + \cdots)</td>
</tr>
<tr>
<td>(\frac{1}{p!}z^p)</td>
<td>(\frac{1}{p!}z^p + \frac{1}{p!}z^pw + \frac{1}{p!}z^pw^2 + \frac{1}{p!}z^pw^3 + \cdots + \frac{1}{p!}z^pw^q + \cdots)</td>
</tr>
</tbody>
</table>

The lemma and the binomial theorem then give us the following calculation which we present first in elliptical notation and then more formally using summation notation.

\[
e^{z+kw} = 1 + (z + w) + \left(1 + \left(\frac{z^2}{2} + zw + \frac{1}{2}w^2\right) + \left(1 + \left(\frac{z^3}{3!} + \frac{1}{2}z^2w + \frac{1}{3!}zw^2 + \frac{1}{2}w^3\right) + \cdots + \frac{z^p}{p!q!}w^{q-k}\right)\right) \cdots \\
= 1 + (z + w) + \frac{1}{2} \left(\frac{z^2}{2} + zw + \frac{1}{2}w^2\right) + \frac{1}{3!} \left(\frac{z^3}{3!} + \frac{1}{2}z^2w + \frac{1}{3!}zw^2 + \frac{1}{2}w^3\right) + \cdots + \frac{1}{p!q!}z^p\frac{w^q}{p!q!}w^{q-k}\right) \cdots \\
= 1 + (z + w) + \frac{1}{2} \left(\frac{z^2}{2} + zw + \frac{1}{2}w^2\right) + \frac{1}{3!} \left(\frac{z^3}{3!} + \frac{1}{2}z^2w + \frac{1}{3!}zw^2 + \frac{1}{2}w^3\right) + \cdots + \frac{1}{p!q!}z^p\frac{w^q}{p!q!}w^{q-k}\right) \cdots \\
= e^{z+kw}.
\]
Equivalently, we can write the same calculation a bit more cleanly, but more abstractly in summation notation.

\[ e^z e^w = \left( \sum_{p=0}^{\infty} \frac{z^p}{p!} \right) \left( \sum_{q=0}^{\infty} \frac{z^q}{q!} \right) = \left( \sum_{k=0}^{\infty} \left( \sum_{p+q=k} \frac{z^p w^q}{p! q!} \right) \right) = \sum_{k=0}^{\infty} \left( \frac{1}{k!} \sum_{p+q=k} \frac{k!}{p! q!} z^p w^q \right) = \sum_{k=0}^{\infty} \left( \frac{1}{k!} (z + w)^k \right) = e^{z+w}. \]

\[ \square \]

8.6. Complex functions of a real variable. A function which takes a real variable \( t \) as input and produces a complex number as output may be written \( f(t) = u(t) + iv(t) \) where \( u \) and \( v \) are ordinary real functions of the real variable, \( t \). The quintessential example of such a function is \( f(\theta) = e^{i\theta} = \cos \theta + i \sin \theta \). In this section, we consider derivatives and antiderivatives of such functions. In particular we derive the expected formulas for the derivatives and antiderivatives of the exponential and trigonometric functions.

8.6.1. Differentiation.

Definition 8.9. If \( f(t) = u(t) + iv(t) \) is a complex function of a real variable, we define the derivative exactly as in the ordinary case, with the exception that our function takes complex values:

\[ f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} . \]

When this limit exists, we say \( f \) is differentiable at \( t \).

We also write \( \frac{df}{dt} \) for \( f'(t) \). All of the usual facts about derivatives hold: The sum rule for \( \frac{d}{dt} (f(t) + g(t)) \), the product rule for \( \frac{d}{dt} (f(t)g(t)) \), the chain rule for \( f(g(t)) \) (when \( g(t) \) is a real function of a real variable), and so on.
Corollary 8.10. Let $f(t) = u(t) + iv(t)$ be a complex function of a real variable. $f$ is differentiable at $t$ if and only if $u$ and $v$ are differentiable at $t$. In this case (when $f$ is differentiable) we have

$$f'(t) = u'(t) + iv'(t).$$

We omit the straightforward proof.

8.6.2. Differentiating $e^{at}$. We could differentiate $e^{at}$ by differentiating $e^t$ and applying the chain rule. We will instead apply the definition of derivative together with Theorem 8.4 as an illustration of how proofs of differentiation formulas and rules are very similar to the ordinary case of real functions of a real variable. (It is also the case that we could obtain this and other differentiation results by term-by-term differentiation of the defining power series. That method works in the interior of the disk of convergence, but we will not prove that fact.)

Proposition 8.11. Let $a$ be any complex number. Then $\frac{d}{dt}e^{at} = ae^{at}$.

Proof. In the following sequence of equalities, $h$ and $t$ denote real variables.

$$\frac{d}{dt}e^{at} = \lim_{h \to 0} \frac{e^{a(t+h)} - e^{at}}{h}$$

$$= \lim_{h \to 0} e^{at} \frac{e^{ah} - 1}{h}$$

$$= e^{at} \lim_{h \to 0} \frac{e^{ah} - 1}{h}$$

$$= e^{at} \lim_{h \to 0} \frac{-1 + 1 + (ah) + \frac{(ah)^2}{2} + \cdots}{h}$$

$$= e^{at} \lim_{h \to 0} \left( a + \frac{a^2}{2}h + \frac{a^3}{3!}h^2 \cdots \right)$$

$$= ae^{at}.\qed$$
The following corollary shows how the usual differentiation formulas for the trigonometric functions follow from the properties of the complex exponential function.

Corollary 8.12. \( \frac{d}{d\theta} \cos \theta = -\sin \theta \) and \( \frac{d}{d\theta} \sin \theta = \cos \theta \).

**Proof.**
\[
\frac{d}{d\theta} \cos \theta + i \frac{d}{d\theta} \sin \theta = \frac{de^{i\theta}}{d\theta} = -\sin \theta + i \cos \theta.
\]
\(\Box\)

Definition 8.13. As with real functions, if \( f(t) \) and \( F(t) \) are complex functions of a real variable we will write \( \int f(t) \, dt = F(t) + C \) to mean exactly that \( F'(t) = f(t) \).

Corollary 8.14. For every nonzero \( a \in \mathbb{C} \) we have
\[
\int e^{at} \, dt = \frac{e^{at}}{a} + C.
\]

**Proof.** This is just Proposition 8.11 restated in terms of antiderivatives. \(\Box\)

Corollary 8.15. For any real numbers \( \alpha, \beta \in \mathbb{R} \) which are not both zero we have
\[
\int e^{at} \cos \beta t \, dt = \frac{e^{at}(\alpha \cos \beta t + \beta \sin \beta t)}{\alpha^2 + \beta^2} + C
\]
\[
\int e^{at} \sin \beta t \, dt = \frac{e^{at}(\alpha \sin \beta t - \beta \cos \beta t)}{\alpha^2 + \beta^2} + C.
\]

**Proof.** This is just Corollary 8.14 with the real and imaginary parts worked out. Since that might not be entirely obvious, we give the calculation, expanding each side of the equation in Corollary 8.14 in turn assuming that \( a = \alpha + i\beta \).

\[
\int e^{at} \, dt = \int e^{at}(\cos \beta t + i \sin \beta t) \, dt = \int e^{at} \cos \beta t \, dt + i \int e^{at} \sin \beta t \, dt.
\]

\[
e^{at} = \frac{e^{at} \cos \beta t + i e^{at} \sin \beta t}{\alpha + i\beta} = \frac{e^{at} \cos \beta t + ie^{at} \sin \beta t}{\alpha + i\beta} \frac{\alpha - i\beta}{\alpha - i\beta}
\]
\[
= \frac{e^{at}(\alpha \cos \beta t + \beta \sin \beta t) + ie^{at}(\alpha \sin \beta t - \beta \cos \beta t)}{\alpha^2 + \beta^2}.
\]
\(\Box\)
In practice, rather than memorizing formulas like this, we just take any problem involving trigonometry and/or exponential functions and restate it entirely in complex exponentials before solving.

Example 8.16. We show how to find \( \int \cos \theta \cos 3\theta \, d\theta \). We have

\[
\cos \theta \cos 3\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{i3\theta} + e^{-i3\theta}}{2} = \frac{1}{4} \left( e^{4i\theta} + e^{i2\theta} + e^{-i2\theta} + e^{-4i\theta} \right) = \frac{1}{2} (\cos 4\theta + \cos 2\theta).
\]

Thus

\[
\int \cos \theta \cos 3\theta \, d\theta = \frac{1}{8} \sin 4\theta + \frac{1}{4} \sin 2\theta + C.
\]

8.7. Exercises. In the following exercises, carry out the indicated tasks by using complex exponentials.

Exercise 8.5. Calculate \( \int \cos 2\theta \cos 3\theta \, d\theta \).

Exercise 8.6. Calculate \( \int \sin^2 \theta \, d\theta \).

Exercise 8.7. Derive the formula

\[
\cos s \cos t = \frac{\cos(s + t) + \cos(s - t)}{2}.
\]

(This formula is no longer considered a common trigonometric formula, however it has a very interesting history. The interested reader should google the word “prosthaphaeresis.”)

8.8. The definition of \( \pi \) and the periodicity of the exponential and trigonometric functions. While the following section may be a bit more theoretical than the rest of these notes, I include it here so that the reader can see a correct logical development of the properties of the trigonometric functions from first principles. We will assume basic facts (such as the intermediate value theorem) without proof.

8.8.1. The definition of \( \pi \). In standard geometric treatments \( \pi \) is defined to be the ratio of the circumference to the diameter of a circle. There is nothing wrong about this definition, but to make it rigorous we need to make the notion of length of a curved segment rigorous. We show below how a completely analytic definition follows from our treatment of the exponential and trigonometric functions. (This treatment follows that of [Car63]. The interested reader should consult that source for additional details.)
Lemma 8.17. There is some \( \theta > 0 \) such that \( \cos \theta = 0 \).

Proof. Recall that it follows immediately from Definition 8.1 that \( \cos 0 = 1 \). In particular that \( \cos 0 \) is positive.

We argue by contradiction, assuming that \( \cos \theta \) is positive for all \( \theta > 0 \) and arguing until we reach a contradiction. It will then follow that our assumption could not have been true—that is to say, we will know that there is some \( \theta \) so that \( \cos \theta \) is zero or negative. By the intermediate value theorem, we will know that in either case there is some value of \( \theta \) for which \( \cos \theta = 0 \) and thus the Lemma will then be proved.

We now proceed with our hypothetical assumption. If \( \cos \theta \) were positive for \( \theta > 0 \), then (because \( \frac{1}{2} \sin \theta = \cos \theta \)) we would see that \( \sin \theta \) must be increasing for \( \theta > 0 \). In particular (because \( \sin 0 = 0 \)), we would have to have \( \sin 1 > 0 \) and also have \( \sin \theta > \sin 1 \) for all \( \theta > 1 \). Now consider the function

\[
f(\theta) = \cos \theta - \cos 1 + (\theta - 1) \sin 1.
\]

It is easy to see that \( f(1) = 0 \) and that \( f'(\theta) = \sin 1 - \sin \theta \). So under our hypothetical assumption, \( f(\theta) \) is negative for all \( \theta \) bigger than 1. The reader may check that \( f(1 + \frac{\cos 1}{\sin 1}) = \cos (1 + \frac{\cos 1}{\sin 1}) \). Thus we have deduced (from our assumption that \( \cos \theta \) is positive for all \( \theta > 0 \)) that some value of \( \cos \theta \) is negative. This is the desired contradiction, so the proof of the lemma is finished.\(^{11}\)

Definition 8.18. Let \( \theta_0 \) be the smallest positive real number\(^{12}\) for which \( \cos \theta_0 = 0 \). We define \( \pi := 2\theta_0 \).

In other words \( \pi/2 \) is the smallest positive value of \( \theta \) for which \( \cos \theta = 0 \). In our treatment this is not just true—it is the definition of \( \pi \).

---

\(^{11}\)The introduction of the function \( f \) was necessary in order to make each step of our argument transparently valid. However, perhaps this masks the basic idea behind the argument which can be phrased informally this way: If \( \cos \theta \) were always positive, then \( \sin \theta \) would always increase, so \( \sin \theta \) would always be positive, so \( \cos \theta \) would always decrease, so \( \cos \theta \) would eventually be negative. The point of the proof above is to make this argument absolutely precise.

\(^{12}\)We have shown that \( \cos \theta = 0 \) has a solution with \( \theta > 0 \). That there is, in fact, a smallest such solution follows from basic facts about continuous functions, the fact that \( \cos \theta \) is continuous, and the fact that \( \cos 0 \neq 0 \). We will leave the details of this part of the argument out of these notes. The interested reader should take our Math 63.
Figure 17. What Lemma 8.19 asserts about \( \cos \theta, \sin \theta \), and their derivatives on the interval \( 0 \leq \theta \leq \frac{\pi}{2} \). The left hand plot shows the graphs of \( \sin \theta \) and \( \cos \theta \) as functions of \( \theta \). The right hand plot shows the trajectory of \( e^{i\theta} = \cos \theta + i\sin \theta \).

8.8.2. The periodicity of the exponential and trigonometric functions. The argument in the proof of Lemma 8.17 which was based on the faulty assumption that \( \cos \theta \) is positive for all \( \theta > 0 \) can now be applied to the fact (true by virtue of Definition 8.18) that \( \cos \theta \) is positive for all \( 0 \leq \theta < \frac{\pi}{2} \).

Lemma 8.19. The real valued function \( \cos \theta \) decreases monotonically from 1 to 0 on the interval \( [0, \frac{\pi}{2}] \) while \( \sin \theta \) increases from 0 to 1 on that interval. The complex valued function \( e^{i\theta} \) traverses the arc of the unit circle in the first quadrant from 1 to \( i \), giving a continuous one-to-one correspondence from \( [0, \frac{\pi}{2}] \subseteq \mathbb{R} \) to \( \{ z \mid |z| = 1, \Re z \geq 0, \text{ and } \Im z \geq 0 \} \subseteq \mathbb{C} \).

Proof. The assertions of the Lemma are displayed in Figure 17. We now follow the argument in Lemma 8.17. Because \( \cos \theta \) is positive for \( 0 \leq \theta < \frac{\pi}{2} \) and because \( \frac{d}{d\theta} \sin \theta = \cos \theta \) we see that \( \sin \theta \) must be increasing for \( 0 \leq \theta < \frac{\pi}{2} \). In particular (because \( \sin 0 = 0 \)), we must have \( \sin \theta > 0 \) for all \( 0 < \theta \leq \frac{\pi}{2} \). It now follows (because \( \frac{d}{d\theta} \cos \theta = -\sin \theta \)) that \( \cos \theta \) is decreasing for \( 0 < \theta < \frac{\pi}{2} \). From the facts \( \sin \frac{\pi}{2} > 0, \cos \frac{\pi}{2} = 0, \) and \( \cos^2 \frac{\pi}{2} + \sin^2 \frac{\pi}{2} = 1 \) we may deduce \( \sin \frac{\pi}{2} = 1 \). We have now proved the assertions in the first sentence of the Lemma. The rest of the Lemma is just a restatement of these facts in terms of \( e^{i\theta} = \cos \theta + i\sin \theta \). \( \square \)
Figure 18. What we have proved about $\cos \theta$, $\sin \theta$, and their derivatives on the interval $0 \leq \theta \leq \frac{\pi}{2}$ after the first step of the proof of Corollary 8.21. The left hand plot shows the graphs of $\sin \theta$ and $\cos \theta$ as functions of $\theta$. The right hand plot shows the trajectory of $e^{i\theta} = \cos \theta + i \sin \theta$.

Note, in particular, that we have the formula

\[ e^{\frac{i\pi}{2}} = i. \]  

Corollary 8.20. For every $\theta \in \mathbb{R}$ we have

\[ \cos \left( \theta + \frac{\pi}{2} \right) = -\sin \theta \]
\[ \sin \left( \theta + \frac{\pi}{2} \right) = \cos \theta. \]

Proof.

\[
\cos \left( \theta + \frac{\pi}{2} \right) + i \sin \left( \theta + \frac{\pi}{2} \right) = e^{i(\theta+\frac{\pi}{2})} = e^{i\theta} e^{i\frac{\pi}{2}} = (\cos \theta + i \sin \theta) i = -\sin \theta + i \cos \theta.
\]

Corollary 8.21. We have $e^{i\theta} = 1$ exactly when $\theta = 2\pi n$ for some $n \in \mathbb{Z}$. The real functions $\sin \theta$ and $\cos \theta$ are both periodic of period $2\pi$. The complex function $e^{i\theta}$ is a continuous one-to-one correspondence between any interval $[\alpha, \alpha + 2\pi) \subseteq \mathbb{R}$ and the unit circle, $\{ z \mid |z| = 1 \} \subseteq \mathbb{C}$.

Proof. If $\theta = 2\pi n$ for some $n \in \mathbb{Z}$ we have

\[ e^{i\theta} = e^{2\pi n} = \left( e^{\frac{\pi i}{2}} \right)^{4n} = i^{4n} = 1. \]
We now argue the converse. We will actually do more than what is claimed, extending the results of Lemma 8.19 to the whole real line. We saw in Subsection 6.5.3 and in Figure 14 that the effect of multiplication by \(i\) is a counterclockwise rotation of the complex plane by one quarter of a whole revolution. Applying this to Lemma 8.19 via Equation (56) (or its equivalent formulation in Corollary 8.20) we see that \(e^{i\theta}\) traverses the arc of the quarter unit circle from \(i\) to \(-1\) as \(\theta\) goes from \(\frac{\pi}{2}\) to \(\pi\). Equivalently, we see that \(\cos \theta\) decreases monotonically from 0 to \(-1\) on the interval \([\frac{\pi}{2}, \pi]\) while \(\sin \theta\) decreases from 1 to 0 on that interval. What we have deduced so far is presented in Figure 18. Repeating the argument twice more extends the pattern to the whole of \([0, 2\pi]\). We see from this that \(e^{i\theta}\) is a continuous one-to-one correspondence between any interval \([0, 2\pi]\) \(\subseteq \mathbb{R}\) and the unit circle, \(\{z \mid |z| = 1\} \subseteq \mathbb{C}\) and that the smallest positive value of \(\theta\) for which \(e^{i\theta} = 1\) is \(\theta = 2\pi\). On the other hand, \(e^{2n\pi i}\) is 1 so \(e^{i(\theta+2\pi n)} = e^{i\theta}\) for every \(\theta\) and every \(n \in \mathbb{Z}\). The Corollary is now proved. \(\square\)

8.8.3. The careful argument. We are now in a position to give a careful version of the vague definition 6.12. The somewhat disconcerting use of the symbol \(=\) in the following definition is so standard that we choose not to avoid it.

Definition 8.22. For \(z \neq 0\) we write \(\arg z = \theta\) when \(e^{i\theta} = z/|z|\).

This only defines \(\arg z\) up to an integral multiple of \(2\pi\); whenever we have \(\arg z = \theta\) we also have \(\arg z = \theta + 2\pi n\) for every \(n \in \mathbb{Z}\). We say \(\theta\) is the argument of \(z\). By virtue of Corollary 8.21, every \(z \neq 0\) has an argument which is unique up to the ambiguity we just admitted.

Now that we have a rigorous definition, we may also make the vague theorem 6.13 rigorous too.

Corollary 8.23. If \(\arg z_1 = \theta_1\) and \(\arg z_2 = \theta_2\), then \(\arg(z_1z_2) = \theta_1 + \theta_2\).

Proof. If \(z_1 = |z_1|e^{i\theta_1}\) and \(z_2 = |z_2|e^{i\theta_2}\) then

\[z_1z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)} = |z_1z_2|e^{i\theta_1}e^{i\theta_2}\]

\(\square\)

We end this subsection with the observation that our discussion gives the usual picture of polar coordinates. If \(z = x + yi\) is any complex number, we can set \(r = |z| = \sqrt{x^2 + y^2}\) and \(\theta = \arg z\). The definition 8.22 of \(\arg\) then gives \(z = re^{i\theta}\) so by the definition 8.1 of \(\sin\) and \(\cos\) we have \(x = r\cos \theta\) and
Figure 19. The usual picture of polar coordinates in terms of the modulus and argument of a complex number

\[ y = r \sin \theta. \] This brings us full circle\(^{13}\) back to the geometric picture of the trigonometric functions.

9. **Linear ODE's with Constant Coefficients Revisited.**

9.1. Generalities. We wish to use complex functions of a real variable to look for real functions of a real variable which satisfy an equation of the form

\[ \alpha f''(t) + \beta f'(t) + \gamma f(t) = 0 \]  

where \( \alpha, \beta, \gamma \in \mathbb{R} \) are real constants and \( t \) is a real independent variable. We also wish to understand associated Boundary and Initial Value problems in this manner. We will do this by finding complex valued solutions to Equation (57) and harvesting our real valued solutions from an understanding of the complex valued solutions. Throughout this section we will assume that our functions are defined and differentiable enough in their domains that all

\(^{13}\)The pun is intentional.
of the formulas make sense. (More precisely, \( I \subseteq \mathbb{R} \) is some open interval, and we are considering the functions with domain \( I \) which have continuous second derivatives throughout \( I \).)

For the following discussion, given a real or complex valued function \( f(t) \) define the function \( Lf(t) \) by the equation

\[
Lf(t) := \alpha f''(t) + \beta f'(t) + \gamma f(t).
\]

\( L \) is a function which takes a function, \( f \), as input and returns a function, \( Lf \), as output. Furthermore, the solutions to Equation (57) are exactly those \( f \) for which \( Lf = 0 \). The following properties of \( L \) follow directly from basic properties of differentiation.

**Proposition 9.1.** If \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( L \) is defined as in Equation (58) then the following equalities hold for all complex valued functions \( f \) and \( g \) and all complex scalars \( a \in \mathbb{C} \).

1) \( L(f + g) = Lf + Lg \).
2) \( L(af) = afL \).
3) \( \overline{Lf} = L\overline{f} \).
4) \( Le^{at} = e^{at}(\alpha a^2 + \beta a + \gamma) \).

**Proof.** The proofs of all the parts are straightforward application of the basic properties of differentiation. We illustrate by giving the details to the argument behind item 3. The arguments for the other items are similar. Write \( f(t) = u(t) + iv(t) \) for real valued functions \( u \) and \( v \). We have

\[
\frac{d}{dt}(\overline{f(t)}) = \frac{d}{dt}(u(t) - iv(t)) = u'(t) - iv'(t) = \overline{f'(t)}.
\]

It follows that

\[
\overline{Lf} = \overline{\alpha f'' + \beta f' + \gamma f} = \overline{\alpha f''} + \overline{\beta f'} + \overline{\gamma f} = \alpha(\overline{f''}) + \beta(\overline{f'}) + \gamma \overline{f} = L\overline{f}.
\]

\( \square \)

**Corollary 9.2.** If \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( L \) is defined as in Equation (58) then the following are true.

1) If \( Lf = 0 \) then \( L(\text{Re } f) = 0 \) and \( L(\text{Im } f) = 0 \).
2) If \( Lf = 0 \) and \( a \in \mathbb{C} \) then \( Laf = 0 \)

**Proof.** If \( Lf = 0 \) then

\[
L(\text{Re } f) = L \left( \frac{f + \overline{f}}{2} \right) = \frac{Lf + \overline{Lf}}{2} = 0.
\]

The other item is a trivial application of Proposition 9.1, part 2. \( \square \)
Part 1 of Corollary 9.2 allows us to harvest real solutions from a complex solution. Part 2 allows us to modify a solution if, for example, we want to satisfy some initial or boundary value conditions.

9.2. Examples.

Example 9.3. We find a real function \( f(t) \) for which

\[
f''(t) + 25 = 0, \quad f(3) = -1, \text{ and } f'(3) = 0.
\]

The characteristic equation is \( a^2 + 25 = 0 \) which has roots \( a = \pm 5i \). Thus \( e^{5it} = \cos(5t) + i\sin(5t) \) is a complex valued solution of the differential equation, but not of the boundary condition. Taking real and imaginary parts, we see that \( \cos 5t \) and \( \sin 5t \) are independent solutions to the differential equation so \( C_1 \cos 5t + C_2 \sin 5t \) is the general solution. We could use the boundary conditions to solve for \( C_1 \) and \( C_2 \). Here we show how to do the same thing by “guess and fudge.” The idea is to find a complex constant \( \rho \) so that the real part of \( \rho e^{5it} \) satisfies the boundary conditions. Corollary 9.2 then tells us that that real part also satisfies the differential equation and so is our final answer. Towards this end, we compute \( (\rho e^{5it})' \bigg|_{t=3} = a5ie^{15i} \).

We see that we can make this purely imaginary by taking \( a = \rho e^{-15i} \) for any real number \( \rho \). Thus \( f(t) = \text{Re} (\rho e^{-15i}e^{5it}) \) satisfies \( f'(3) = 0 \). To satisfy the other boundary condition, we need \( f(3) = \text{Re} (\rho e^{-15i}e^{5it}) \bigg|_{t=3} = \rho \) to be equal to \(-1\). So if we take \( \rho = -1 \), our \( f(t) \) will satisfy the differential equation and both of the boundary conditions. To complete this work, we simplify to put the answer in a usual form.

\[
f(t) = \text{Re} \left( -e^{-15i}e^{5it} \right) = -\text{Re} \left( e^{(5t-15)i} \right) = -\cos(5t - 15)
\]

Alternatively, if one wishes to find the \( C_1 \) and \( C_2 \) we could calculate

\[
f(t) = \text{Re} \left( -e^{-15i}e^{5it} \right) = -\text{Re} (\{\cos(-15) + i\sin(-15)\} \{\cos(5t) + i\sin(5t)\})
= -\cos(-15)\cos(5t) + \sin(-15)\sin(5t)
= -\cos(15)\cos(5t) - \sin(15)\sin(5t).
\]

Thus, \( C_1 = -\cos(15) \) and \( C_2 = \sin(15) \).

Example 9.4. We find a real function \( f(t) \) for which

\[
f''(t) + 4f'(t) + 5f(t) = 0, \quad f(0) = 2, \text{ and } f'(0) = f''(0).
\]

The characteristic equation is \( a^2 + 4a + 5 = 0 \) which has roots \( a = -2 \pm i \). Thus \( e^{(-2 + i)t} \) is a complex solution of the differential equation, but not of the
boundary condition. To make the homogeneous boundary condition more explicitly homogeneous, we write it as \( f'(0) - f''(0) = 0 \). Now we have

\[
\left[ \left( e^{-2 + i t} \right)' - \left( e^{-2 + i t} \right)'' \right]_{t=0} = (-2 + i) - (-2 + i)^2 = -5 + 5i.
\]

Therefore the real part of \( e^{(-2 + i t)(-5 - 5i)} = (5 - 5i) e^{(-2 + i t)} \) satisfies the homogeneous boundary conditions. Since \( [(5 - 5i) e^{(-2 + i t)}]_{t=0} = 5 - 5i \), the real part of \( \frac{2}{5}(5 - 5i) e^{(-2 + i t)} = (2 - 2i) e^{(-2 + i t)} \) satisfies both the differential equation and the initial value conditions. To complete this work, we simplify to make the answer explicitly in the usual form.

\[
\text{Re}(2 - 2i) e^{(-2 + i t)} = 2 e^{-2t} \text{Re} ((1 - i)(\cos t + i \sin t)) = 2 e^{-2t}(\cos t + \sin t).
\]

9.3. Phase shift revisited. As these examples suggest, complex arithmetic helps to explain the formulas in the box near the end of Section 11.10 of [HHGM+05].

**Proposition 9.5.** Let \( \omega \in \mathbb{R} \) be a specified real number. Each of the following formulas with a pair of additional parameters specifies the same family of functions.

\[
\begin{align*}
(59) \quad C_1 \cos \omega t + C_2 \sin \omega t & \quad C_1, C_2 \in \mathbb{R} \\
(60) \quad A \cos(\omega t + \phi) & \quad A, \phi \in \mathbb{R} \\
(61) \quad A \cos(\omega (t - t_1)) & \quad A, t_1 \in \mathbb{R}
\end{align*}
\]

Before we give the proof of Proposition 9.5, we make a few remarks. The family of functions which is described by the three formulas is the set of solutions of \( y'' + \omega^2 y = 0 \), but we needn’t know that fact to understand the proposition or the proof. As we will see in the proof, the parameters in the various formulas are related by the following equations.

\[
\phi = -t_1 \omega, \quad C_1 = A \cos \phi, \quad C_2 = -A \sin \phi
\]

The proof can (and usually is) accomplished by application of the formula for \( \cos(\alpha + \beta) \), however the complex arithmetic argument given below clarifies the appearance of the polar coordinate formulas in the relations between the parameters. In formulas (60) and (61), we could just as easily used sines instead of cosines.
**Proof of Proposition 9.5.** Let \((A, \phi)\) be the polar coordinates of the point on the plane which has rectangular coordinates \((C_1, -C_2)\). Thus we have \(C_1 - iC_2 = Ae^{i\phi}\). This allows the following calculation.

\[
C_1 \cos \omega t + C_2 \sin \omega t = \text{Re} \left( C_1 e^{i\omega t} - C_2 ie^{i\omega t} \right) = \text{Re} \left( (C_1 - iC_2) e^{i\omega t} \right) = \text{Re} \left( A e^{i(\omega t + \phi)} \right) = A \cos(\omega t + \phi)
\]

Thus, we have established the equivalence of the families given by formula (59) and formula (60). The equivalence of the families given by formula (60) and formula (61) is made clear by the substitution \(\phi = -t_1 \omega\).

**9.4. Exercises.** In each of the following problems, solve the given differential equation, boundary value problem, or initial value problem. Use the language of complex variables wherever it is appropriate.

**Exercise 9.1.** \(y'' - 2y' + 10y = 0\).

**Exercise 9.2.** \(2y'' + 2y' + y = 0\).

**Exercise 9.3.** \(9y'' + 6y' + 82y = 0\), with \(y(0) = -1\) and \(y'(0) = 2\).

**Exercise 9.4.** Consider the differential equation

\(y'' - 2y' + 5y = x + 1\).

a) Write down the associated homogeneous equation and find a complex solution to it.
b) Write down the general real solution to the associated homogeneous equation.
c) Find a particular solution to the original equation using the method of intelligent guessing.
d) Write down the general solution to the original equation.

**10. Fourier Series Revisited**

In this section, we will restate what we know about the Fourier series of a real function of a real variable in terms of complex functions of a real variable. We replace the trigonometric terms in the Fourier series with exponential terms by means of the equations

\[
\cos \theta := \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta := -i\frac{e^{i\theta} - e^{-i\theta}}{2}.
\]
10.1 Reformulation in complex terms. Recall that the Fourier series for \( f(x) \) on the interval \([−\pi, \pi]\) is the series

\[
f(x) \sim a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots
\]

\[
= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)
\]

where \(a_0\) and the \(a_k\) and \(b_k\) are given by the following formulas.

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx
\]

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx
\]

and \( \sim \) is equality on \([−\pi, \pi]\) when \( f(x) \) is continuous and equality at points of continuity when \( f(x) \) is piecewise continuous.

In order to see what Formula (63) looks like in terms of complex exponentials, we make the substitutions given in Equation (62). We first do the substitution in just one term of the sum:

\[
a_k \cos kx + b_k \sin kx = a_k \frac{e^{ikx} + e^{-ikx}}{2} - b_k i \frac{e^{ikx} - e^{-ikx}}{2} = \frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx}.
\]

From this we see that if we set

\[
c_0 = a_0, \quad c_k = \frac{a_k - ib_k}{2}, \quad \text{and} \quad c_{-k} = \frac{a_k + ib_k}{2}
\]

we have

\[
a_k \cos kx + b_k \sin kx = c_k e^{ikx} + c_{-k} e^{-ikx}
\]

for all positive values of \( k \) and \( a_0 = c_0 e^{i0x} \). Thus, we can re-express the formula (63) as

\[
f(x) \sim c_0 e^{i0x} + c_1 e^{i1x} + c_2 e^{i2x} + \cdots + c_{-1} e^{i(-1)x} + c_{-2} e^{i(-2)x} + \cdots
\]

\[
= \sum_{k=-\infty}^{\infty} c_k e^{ikx}
\]
10.2. The complex orthogonality relations and the integral formula for the \(c_k\). The form the orthogonality relations take for complex exponentials is presented in the following lemma.

Lemma 10.1. If \(k\) and \(\ell\) are any two integers, then

\[
\int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x} \, dx = \begin{cases} 
0 & \text{if } k \neq \ell, \\
2\pi & \text{if } k = \ell.
\end{cases}
\]

Proof. If \(k \neq \ell\),

\[
\int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x} \, dx = \int_{-\pi}^{\pi} e^{i(k-\ell)x} \, dx = \left[ \frac{e^{i(k-\ell)x}}{i(k-\ell)} \right]_{-\pi}^{\pi} = 0
\]

because \(e^{i(k-\ell)x}\) has period \(\frac{2\pi}{k-\ell}\) which evenly divides \(2\pi = \pi - (-\pi)\). On the other hand, if \(k = \ell\), we have

\[
\int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi.
\]

\[\square\]

The orthogonality relations allow a proof of the formulas listed at (64)–(66) in the following unified and simplified form.

Corollary 10.2. In Formula (68) we have

\[c_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\ell x} \, dx.\]

Sketch of proof.

\[
\int_{-\pi}^{\pi} f(x)e^{-i\ell x} \, dx = \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} c_k e^{ikx} \right) e^{-i\ell x} \, dx
\]

\[
= \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} c_k e^{ikx} e^{-i\ell x} \right) \, dx
\]

\[
\sum_{k=-\infty}^{\infty} \left( c_k \int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x} \, dx \right)
\]

\[
= \cdots + 0 + 0 + 0 + c_\ell 2\pi + 0 + 0 + 0 + \cdots
\]

\[
= 2\pi c_\ell.
\]

Each of the above equalities uses basic facts about infinite sums, none of which we have proved carefully. The only one which requires comment is
the one labeled Equation (69) where we move the integral inside the sum. Such a manipulation is clearly true for a finite sum. For an infinite sum an argument is necessary, which we explicitly omit, but can be found in the references.

10.3. Examples. Here is an example of the application of the formula for the coefficients given in Corollary 10.2.

Example 10.3. Suppose that

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{otherwise}. \end{cases}$$

We then have

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx = \frac{1}{2\pi} \int_{0}^{\pi} e^{-ikx} \, dx = \begin{cases} \frac{1}{2} & \text{if } k = 0, \\ \left[ \frac{1}{2\pi} \cdot \frac{e^{-ik\pi}}{1} \right]_0^\pi & \text{otherwise}. \end{cases}$$

Furthermore, when $k \neq 0$ we have

$$\left[ \frac{1}{2\pi} \cdot \frac{e^{-ik\pi}}{1} \right]_0^\pi = \frac{i}{2\pi k} (e^{-ik\pi} - 1) = \frac{i}{2\pi k} \left( (e^{-i\pi})^k - 1 \right)$$

$$= \frac{i}{2\pi k} \left( (-1)^k - 1 \right) = \begin{cases} 0 & \text{k is even,} \\ \frac{i}{2\pi k} (-2) & \text{otherwise}. \end{cases}$$

Thus, in light of the equations (67) we get $a_0 = c_0 = \frac{1}{2}$, $a_k = 0$ for $k > 0$, $b_k = 0$ for $k$ even and $b_k = \frac{2}{2\pi k}$ for $k$ odd. That is to say we have

$$f \sim \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \cdots.$$

10.4. Exercises.

Exercise 10.1. Invert the formula at (67) to give formula for $a_k$ and $b_k$ in terms of $c_k$.

Exercise 10.2. Use the methods of Example 10.3 to find the Fourier series for $f(x) = x$ on the interval from $[-\pi, \pi]$.

Exercise 10.3. Express the Fourier series

$$\frac{1}{2} - \frac{1}{3} \sin x + \frac{1}{4} \sin 2x - \frac{1}{5} \sin 3x + \cdots$$

as a complex Fourier series—that is to say, find the $c_k$ in Equation (68) explicitly for this series and write it out as an elliptical expression with two ellipses.
11. List of Properties of the Exponential Function Which We Have Proved and/or Used

Here is a list of the properties of the exponential function which we have used or proved. The list is redundant; many of the properties follow from the first couple. In general, the more fundamental properties are listed first. Here, \( z, w \in \mathbb{C} \) and \( \alpha, \beta, \theta \in \mathbb{R} \).

\[
\begin{align*}
  e^0 &= 1 \\
  e^z &= e^z \\
  e^{z+w} &= e^z e^w \\
  e^{i\theta} &= \cos \theta + i \sin \theta \\
  \frac{de^{at}}{dt} &= a e^{at} \\
  e^{\frac{a\theta}{i}} &= i
\end{align*}
\]

12. Solutions to Selected Exercises.

Solution to Exercise 2.1. a) Separation of variables gives \( \frac{dy}{a-by} = dt \). Integration gives \( -\frac{1}{b} \ln |a-by| = t + \mathcal{C} \). Exponentiation gives \( a-by = Ae^{-bt} \), which yields the answer

\[
y = \frac{a}{b} + Be^{-bt},
\]

where \( a \) and \( b \) are the constants specified in the problem and \( B \) is a constant we may choose arbitrarily to satisfy an initial condition.

b) Plugging the desired values into the general solution gives \( 0 = \frac{a}{b} + B \), so \( B = -\frac{a}{b} \). Thus the solution we seek is

\[
y = \frac{a}{b} - \frac{a}{b} e^{-bt}.
\]

c) Yes, there is an equilibrium solution. Looking at the original equation we see that if \( y \) is a constant solution, then \( 0 = a-by \) so \( y = \frac{a}{b} \). We may also see this constant solution as a special case of the general solution where \( B = 0 \).

\[\square\]

Solution to Exercise 3.1. The given equation, \( y'-3y = e^{2x} \), is in standard form. We have \( p(x) = -3 \), so \( P(x) = -3x \) and our integrating factor is \( e^{-3x} \).
Following the trick gives the following sequence:
\[ e^{-3x}y' - 3e^{-3x}y = e^{-x} \]
\[ \int d\left(e^{-3x}y\right) = \int e^{-x} \, dx \]
\[ e^{-3x}y = -e^{-x} + C \]
\[ y = -e^{2x} + Ce^{3x}. \]

One should check this solution by differentiating and seeing that for this \( y \), we do indeed have \( y' - 3y = e^{2x} \) as claimed. \( \square \)

**Solution to Exercise 3.2.** \( y' + y = e^{-x} \) is in standard form. We have \( p(x) = 1 \), so \( P(x) = x \) and our integrating factor is \( e^x \). Following the trick gives the following sequence:
\[ e^xy' + ye^x = 1 \]
\[ \int d\left(e^xy\right) = \int dx \]
\[ e^xy = x + C \]
\[ y = xe^{-x} + e^xC. \]

When \( x = 0 \), \( y = C \), so to get \( y|_{x=0} = 0 \) we take \( C = 0 \). Thus, our solution is \( y = xe^{-x} \). Again one should check by differentiating and plugging into the original equation. \( \square \)

**Solution to Exercise 3.3.** \( y' = 100 - y \) in standard form becomes \( y' + y = 100 \), so \( P(x) = x \) and our integrating factor is \( e^x \). Following the usual pattern we get the following sequence:
\[ \int d\left(e^xy\right) = \int 100e^x \, dx \]
\[ e^xy = 100e^x + C \]
\[ y = 100 + Ce^{-x}. \]

Again, one should check that the computed result satisfies the original equation. \( \square \)

**Solution to Exercise 3.4.** To find the integrating factor, we first write our equation in standard form:
\[ y' + \frac{1-x}{x}y = \frac{e^{2x}}{x}. \]
We then use the coefficient of $y$ to find the integrating factor as follows.

\[
ed \int \frac{1}{x} \, dx = e^{\ln x} = xe^{-x}.
\]

(A finer point which you did not need to include in your answer: We have ignored possible constants and assumed $x$ positive in finding this. Since we will check that the integrating factor works as we use it, this nonchalance will not hurt us.) Now that we have the integrating factor we multiply the \textit{standard form} of our equation by it to get $y'xe^{-x} + (1 - x)ye^{-x} = e^x$. (We could also have arrived at this point by multiplying the original form of the differential equation by $e^{-x}$. If we did so, we would still be obligated to explain where the $e^{-x}$ came from.) Recognizing the left hand side of this most recent form of the equation as the derivative of a product now allows us to finish the problem by the following calculation.

\[
d \left(yxe^{-x}\right) = e^x \\
yxe^{-x} = \int e^x \, dx \\
yxe^{-x} = e^x + C \\
y = \frac{e^{2x} + Ce^x}{x}.
\]

(Another finer point: These solutions are valid when $x \neq 0$, so our argument yields solutions in the intervals $(-\infty, 0)$ and $0, \infty$. One can mix and match these to get solutions defined everywhere except $x = 0$.)

\begin{solution}{exercise 3.6}
The integrating factor is $e^{\int \tan x \, dx} = e^{\ln |\sec x|} = |\sec x|$. Since we are in a range where $\sec x > 0$ we may omit the absolute value signs. Multiplying through by $\sec x$ gives

\[
dy \sec x + y \sec x \tan x = 0.
\]

Recognizing the left hand side as a derivative of the product $y \sec x$ we see that our equation amounts to insisting that $y \sec x$ is constant. That is to say $y = A \cos x$ as desired.

\begin{solution}{exercise 3.5}
$xy' + (1 - x)y = \frac{e^{2x}}{x}$ in standard form becomes $y' + \frac{1}{x}y = \frac{e^{2x}}{x^2}$. Notice that $p(x)$ and $q(x)$ are discontinuous at $x = 0$, so we should only try to solve this equation on intervals which do not contain 0. We'll assume that $x > 0$. The $x < 0$ case is similar. We have $p(x) = \frac{1}{x} - 1$ so

\[
y = \frac{e^{2x}}{x^2} + Ce^{2x} + 2C.
\]
\[ P(x) = \ln x - x \] and our integrating factor is \( \exp(\ln x - x) = xe^{-x} \). Following
the usual pattern gives:

\[
y'xe^{-x} + (1 - x)e^{-x}y = \frac{e^x}{x}
\]

\[
\int d\left( yxe^{-x} \right) = \int \frac{e^x}{x} \, dx.
\]

Antidifferentiating \( \frac{e^x}{x} \) is a problem—in fact, there is no closed form solution
in terms of elementary functions. The usual dodge in this situation is to define

\[
f(x) := \int_1^x \frac{e^t}{t} \, dt \quad x > 0.
\]

The second fundamental theorem of calculus then gives us \( f'(x) = \frac{e^x}{x} \) for
\( x > 0 \). \( f(x) \) may be computed by numerical integration when needed. We
then may continue as follows:

\[
yxe^{-x} = f(x) + C
\]

\[
y = \frac{e^x}{x} f(x) + C \frac{e^x}{x}.
\]

Once again, one should check by plugging into the original equation. \( \Box \)

**Solution to Exercise 4.1.** In Equation (25) \( k \) is positive, so we see that
whenever the graph of \( H \) is below that of \( M \)—that is to say, whenever \( H - M \)
is negative—we have \( dH/dt \) is positive—that is to say, \( H \) is increasing. \( \Box \)

**Solution to Exercise 4.2.** Under the given conditions, Equation (25) becomes

\[
\frac{dH}{dt} = -k(H(t) - Ae^{\ell t})
\]

or, in standard form

\[
H' + kH = kAe^{-\ell t}.
\]

our integrating factor is \( e^{kt} \) and the usual method yields

\[
d\left( e^{kt}H \right) = kAe^{(k-\ell) t} \, dt
\]

\[
e^{kt}H = \frac{k}{k-\ell} Ae^{k-\ell t} + C
\]

\[
H = \frac{k}{k-\ell} Ae^{-\ell t} + Ce^{-kt}.
\]

(At least as long as \( \ell \neq k \). At \( t = 0 \) we get \( H(0) = \frac{k}{k-\ell}A + C \), so
\( C = H(0) - \frac{k}{k-\ell}A \). It is physically reasonable to assume that \( \ell < k \). This
Figure 20. Solutions to $H' = -k(H - Ae^{-\ell t})$ for $k = 1.0$, $A = 3.0$ and $\ell = 0.5$.

The calculation becomes:

$$d \left( e^{kt}H \right) = kA \, dt$$

$$e^{kt}H = tkA + C$$

$$H = tkAe^{-kt} + Ce^{-kt}.$$

Evaluating at $t = 0$ gives $C = H(0)$. 

Solution to Exercise 5.1. The associated homogeneous equation is $y'' + y' + y = 0$ which has characteristic equation $r^2 + r + 1 = 0$. The roots of this characteristic equation are $r = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus the general solution to the homogeneous equation is

$$y_{\text{homo}} = e^{-\frac{1}{2}x} \left( C_1 \cos \left( \frac{\sqrt{3}}{2}x \right) + C_2 \sin \left( \frac{\sqrt{3}}{2}x \right) \right).$$
Intelligent guessing gives a particular solution \( y_{\text{part}} = 17 \). Thus, the general solution to the original equation is given by

\[
y_{\text{homo}} + y_{\text{part}} = e^{-\frac{3}{2}x} \left( C_1 \cos \left( \frac{\sqrt{3}}{2}x \right) + C_2 \sin \left( \frac{\sqrt{3}}{2}x \right) \right) + 17
\]

\[\□\]

Solution to Exercise 5.2. A particular solution to the equation

\[
y'' + by' + cy = R
\]

where \( R \) is a constant is given by the constant function \( y = R/c \). This means that solutions to Equation (70) are of the form

\[
y + R/c
\]

where \( y \) is a solution to the associated homogeneous equation

\[
y'' + by' + cy = 0.
\]

If we think of Equation (70) as representing some physical system experiencing a constant external force, this tells us that the physical system behaves just as the system with no force (modeled by Equation (71)) except that there is a displacement to a new rest position. \[\□\]

Solution to Exercise 5.3. We find the general solution to the equation

\[
y'' + y' - 6y = \cos x.
\]

by means of the following steps:

a) First we find the general solution to the associated homogeneous equation. \( y = e^{rx} \) satisfies \( y'' + y' - 6y = 0 \) exactly when \( r \) satisfies \( r^2 + r - 6 = (r + 3)(r - 2) = 0 \). Thus \( r = -3 \) or \( r = 2 \). Therefore the general solution to the associated homogeneous equation is \( y = C_1 e^{-3x} + C_2 e^{2x} \).

b) Next we find any particular solution to the given equation, (72). Looking at the form of the right hand side, we guess a solution of the form \( A \cos x + B \sin x \). Pursuing this guess, we get

\[
y = A \cos x + B \sin x
\]

\[
y' = B \cos x - A \sin x
\]

\[
y'' = -A \cos x - B \cos x,
\]
so $y'' + y' - 6y = (-7A + B)\cos x + (-A - 7B)\sin x$. Therefore, our guess is a solution to the original differential equation, (72), when $A$ and $B$ satisfy the pair of equations

$$
-7A + B = 1 \\
A + 7B = 0.
$$

The solution to these equations is $A = \frac{7}{50}$ and $B = \frac{1}{50}$. Thus the particular solution to the original differential equation we find is $y = \frac{7}{50}\cos x + \frac{1}{50}\sin x$.

c) We finish the problem by using the work from the previous parts. The general solution to the original equation, (72), is the sum of the general solution to the associated homogeneous equation and a particular solution to the original equation. Therefore the general solution to the original problem is

$$
y = C_1 e^{-3x} + C_2 e^{2x} + \frac{-7}{50}\cos x + \frac{1}{50}\sin x.
$$

\[\square\]

*Solution to Exercise 5.4.* We know that the general solution to

$$
y'' - y = 0
$$

is

$$
y = C_1 e^x + C_2 e^{-x}.
$$

To get a particular solution to

$$
y'' - y = x^2
$$

it is reasonable to guess that $y$ should be a polynomial. Furthermore, degree two may well be enough, so we try that: Plug $y = D_2 x^2 + D_1 x + D_0$ into Equation (73) to get

$$
2D_2 - D_2 x^2 - D_1 x - D_0 = x^2.
$$

Equating like powers of $x$ we get the three equations

$$
2D_2 - D_0 = 0 \\
-D_1 = 0 \\
-D_2 = 1
$$


which are easily seen to have the solution \( D_2 = -1, D_1 = 0, D_0 = -2 \). 
So our particular solution is \( y = -x^2 - 2 \) which is readily seen to satisfy 
Equation (73). Thus the general solution to Equation (73) is given by

\[
y = C_1 e^x + C_2 e^{-x} - x^2 - 3.
\]

\[\square\]

**Solution to Exercise 5.5.** We continue the discussion of Equation (30) given 
in the problem and the associated text, using the notation introduce there. 
We have already deduced that the solution to the associated initial value problem is

\[
s(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{A/m}{\omega^2 - \eta^2} \sin(\eta t).
\]

Where \( C_1 = s_0 \) and \( C_2 \) is determined by the equation

\[\nu_0 = C_2 \omega + \frac{\eta A/m}{\omega^2 - \eta^2}.
\]

In this problem we are asked to set \( s_0 = \nu_0 = 0 \) so we have

\[
C_2 = \frac{\eta A/(m \omega)}{\eta^2 - \omega^2}
\]

and our solution is

\[
s(t) = \frac{\eta A/(m \omega)}{\eta^2 - \omega^2} \sin(\omega t) + \frac{A/m}{\omega^2 - \eta^2} \sin(\eta t).
\]

\[\square\]

**Solution to Exercise 6.1.** First of all, suppose that \( x y_1 = x y_2 = 1 \). Then 
the listed properties allow the following deduction that \( y_1 = y_2 \).

\[
y_2 = 1 \cdot y_2 = (x y_1) y_2 = x(y_1 y_2) = x(y_2 y_1) = (x y_2) y_1 = 1 \cdot y_1 = y_1.
\]

Thus, we may improve on the property listed on the right side of (44): For 
every real number \( x \neq 0 \) there is exactly one real number \( y \) such that \( x y = 1 \). 
We define \( 1/x \) to be this \( y \). We define \( x_1/x_2 := x_1 \cdot (1/x_2) \).

\[\square\]

**Solution to Exercise 6.2.** a) \( (1 + 2i) + (3 + 4i) = 4 + 6i \).
b) \( (1 + 2i)(3 + 4i) = -5 + 10i \).
c) \( \frac{1 + 2i}{3 + 4i} = \frac{11 + 2i}{25} \).
d) This is the only part of this problem which should have required a little thought. We can arrive at the answer in the following way. We have $a + bi = \sqrt{3} + 4i$ exactly when $(a + bi)^2 = 3 + 4i$. Comparing real and imaginary parts gives the pair of equations
\[
\begin{align*}
a^2 - b^2 &= 3 \\
2ab &= 4.
\end{align*}
\]

The second equation yields $b = 2/a$ which we can plug into the first getting $a^4 - 3a^2 - 4 = 0$ which is a quadratic expression in $a^2$. Since $a^4 - 3a^2 - 4 = (a^2 - 4)(a^2 + 1)$ and $a$ is real, we deduce that $a = \pm 2$. It follows that $b = \pm 1$. Thus $\sqrt{3 + 4i} = \pm(2 + i)$. One should double check this answer by verifying that $(2 + i)^2 = 3 + 4i$.

\[\square\]

**Solution to Exercise 6.3.**

i) \[\frac{1+i}{2+i} = \frac{(1+i)(2-i)}{(2+i)(2-i)} = \frac{3}{5} = i = 0 + 1 \cdot i.\]

ii) \[\frac{4+3i}{1+i} = \frac{(4+3i)(1-i)}{(1+i)(1-i)} = \frac{11+2i}{2} = \frac{11}{2} + \frac{1}{2}i.\]

iii) \[\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\]
\[= \frac{1}{4} + \frac{3}{4} = 1 = 1 + 0i.\]

\[\square\]

**Solution to Exercise 6.4.** Express $\sqrt{5+12i}$ as $a+bi$ by solving the equation $(a+bi)^2 = 5 + 12i$. (In the problems as handed out, the second “12” was mistyped as “21.” I hope that this did not cause undue confusion.) The equation we need to solve is $5 + 12i = (a^2 - b^2) + (2ab)i$. This is two quadratic equations in two unknowns:
\[
\begin{align*}
a^2 - b^2 &= 5 \\
2ab &= 6.
\end{align*}
\]

We are looking for $a, b \in \mathbb{R}$ which satisfy these equations. In general, such systems can be hard, but if we look for integral solutions, we can find integral solutions of this system by inspection. If $(a, b)$ is a solution, so is $(-a, -b)$, so we may as well assume $a > 0$. Clearly we must have $a > 2$. Trying $a = 3$ gives $b = 2$. Thus $\sqrt{5+12i} = \pm(3 + 2i)$.

\[\square\]

**Solution to Exercise 6.5.** In this problem I asked, “Is there any power of $1+2i$ which is equal to 1? Why or why not?” Of course, one could correctly
answer \((1 + 2i)^0 = 1\), and I would have to count that as correct. However I intended to ask the question “Is \((1 + 2i)^n = 1\) for any positive integer \(n\)?” (I’ve now changed that in the body of these notes)

**Solution to Exercise 6.6.** Is there any power of \(\frac{3 + 4i}{5}\) which is equal to 1? Why or why not? Again, the zeroth power answers the stated problem. Again, I’ll assign this problem again next week.

**Solution to Exercise 6.8.** The modulus is \(|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}\). The argument is half of a right angle—that is to say \(\frac{\pi}{4}\). Of course adding any multiple of \(2\pi\) to this also gives a valid value of the argument.

**Solution to Exercise 6.9.** To multiply complex numbers in polar (modulus and argument) form, we “multiply the moduli and add the arguments.” Thus, to square a number we square the modulus and double the argument. Thus, to take the square root, we take the square root of the modulus and halve the argument. So our answer is \(|\sqrt{1+i}| = \sqrt{\sqrt{2}} = \sqrt[4]{2}\) and \(\text{arg} \sqrt{1+i} = \frac{\pi}{4}\). The alert student will notice that if we write \(\text{arg}(1+i) = (2 + \frac{1}{4})\pi\) we get \(\text{arg} \sqrt{1+i} = \frac{3\pi}{8}\) which gives a different square root. Not to worry though, adding \(\pi\) to the argument multiplies the number by \(-1\), so if \(a\) is a square root of \(\sqrt{1+i}\), so is \(-a\). In other words, we have two square roots with two arguments—separated by an angle of \(\pi\). (I will give full credit for the student who only uses one square root, but give an extra credit point for getting both.)

**Solution to Exercise 6.11.** To prove that for every \(w, z \in \mathbb{C}\) we have the equality

\[|z + w|^2 = |z - w|^2 = 2|z|^2 + 2|w|^2\]

the left hand side and see that we get the right:

\[|z + w|^2 + |z - w|^2 = (z + w)(\overline{z + w}) + (z - w)(\overline{z - w}) = z\overline{z} + w\overline{w} + z\overline{w} + z\overline{z} - w\overline{w} - z\overline{w} + w\overline{w} = 2z\overline{z} + 2w\overline{w} = 2|z|^2 + 2|w|^2\]

Interpreted geometrically, this result says that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides of the parallelogram.

**Solution to Exercise 6.12.** There is no positive power of \(1 + 2i\) which is equal to 1. This is because \(|1 + 2i|^n = |1 + 2i|^n = (\sqrt{5})^n > 1\) for every positive \(n\). Unfortunately, this argument fails for \(\frac{3 + 4i}{5}\) because \(\frac{3 + 4i}{5}\) which is equal to 1. Here is one way to solve this problem—perhaps you can find
a simpler way. If \( \left( \frac{3 + 4i}{5} \right)^n = 1 \) then \( (3 + 4i)^n = 5^n \). However, if we look at remainders modulo 5, we can see that this will never happen. More specifically we prove the following proposition by induction on \( n \): For every \( n \geq 1 \), \( (3+4i)^n = 3+4i+5(z_n) \) where \( z_n \) is a complex number with integral real and imaginary parts. The assertion is clearly true for \( n = 1 \)—we just take \( z_1 = 0 \). We now show that the \( n \)-th case follows from the \( n-1 \)st. If

\[
(3 + 4i)^{n-1} = 3 + 4i + 5z_{n-1}
\]

then

\[
(3 + 4i)^n = (3 + 4i)(3 + 4i + 5z_{n-1}) = (3 + 4i)^2 + 5(3 + 4i)z_{n-1}
\]

\[
= -7 + 24i + 5(3 + 4i)z_{n-1} = 3 + 4i + -10 + 20i + 5(3 + 4i)z_{n-1}
\]

\[
= 3 + 4i + 5 ((3 + 4i)z_{n-1} - 2 + 4i).
\]

So we may take \( z_n = (3 + 4i)z_{n-1} - 2 + 4i \). Certainly, the parts of \( z_n \) are integers if the parts of \( z_{n-1} \) were integers. Thus every power of \( (3 + 4i) \) is “equal to \( (3 + 4i) \) modulo 5” and no power of \( (3 + 4i) \) could possibly be a power of 5.

\[ \square \]

**Solution to Exercise 6.14.** In our solutions, we redraw the original figure with a solid line and indicate the modified one with a dashed line.

a)
b) 

\[ \begin{align*} 
-2 & \quad -1.5 & \quad -1 & \quad -0.5 & \quad 0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 \\
-2 & \quad -1.5 & \quad -1 & \quad -0.5 & \quad 0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 
\end{align*} \]

\[
\begin{array}{c}
\text{2}
\end{array}
\]

c) 

\[ \begin{align*} 
-2 & \quad -1.5 & \quad -1 & \quad -0.5 & \quad 0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 \\
-2 & \quad -1.5 & \quad -1 & \quad -0.5 & \quad 0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 
\end{align*} \]

\[
\begin{array}{c}
\text{2}
\end{array}
\]
Here, perhaps the answer looks confusing because the circle running from 0 to the point marked $a^{-1}$ joins with the circle joining $a^{-1}$ to the angle. The two circles have different radii and different centers. At any rate, hopefully the added marking makes things clearer.

**Solution to Exercise 7.1.** The associated real series is $\sum_{n=0}^{\infty} \frac{r^n}{n+1}$. We use the ratio test to determine for which $r$ this series is absolutely convergent. The relevant limit is

$$L = \lim_{n \to \infty} \frac{|r|^{n+1}}{|r|^n} \cdot \frac{1 + n^2}{1 + (n + 1)^2} = |r| \lim_{n \to \infty} \frac{1 + n^2}{1 + (n + 1)^2} = |r| \lim_{n \to \infty} \frac{1}{n^2} \left( \frac{1}{\frac{1}{n}} + \frac{1}{n+1} \right)^2 = |r|.$$ 

Thus when $|r| < 1$ we have absolute convergence and when $|r| > 1$ we have divergence. In other words the radius of convergence is 1. By Fact 7.5, the disk of absolute convergence is $\{z \mid |z| < 1\}$. 

**Solution to Exercise 8.1.** a) $e^{\pi i/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$.

b) $3e^{\pi i} = 3(\cos \pi + i \sin \pi) = -3$.

c) $e^{\pi i/4} - e^{-\pi i/4} = 2i \text{Im}(e^{\pi i/4}) = 2i \sin \frac{\pi}{4} = i \sqrt{2}$. 

$\square$
Solution to Exercise 8.2. We use $e^{x+y} = e^x e^y$ to prove that $e^z \neq 0$ for all $z$. First note that if $a \in \mathbb{C}$ then $a0 = a(0 - 0) = a0 - a0 = 0$. Thus if $ab \neq 0$ then neither $a$ nor $b$ is zero. But $e^{x} e^{-x} = e^0 = 1 \neq 0$ so $e^x \neq 0$. (Other proofs are possible, but I think that this is the shortest one that flows only from our axioms.)

Solution to Exercise 8.3. Find all those complex numbers $z$ for which $e^z = 1$. Write $a = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Then $|e^z| = \sqrt{e^{2x}} = \sqrt{e^x e^y} = e^{2\alpha} = e^\alpha$, so for $e^z$ to be 1, we must have $\alpha = 0$. Now using $e^{i\beta} = \cos \beta + i\sin \beta$ we see that $\beta$ must be a multiple of $2\pi$. Thus $e^z = 1$ exactly when $z = 2\pi n$ for some $n \in \mathbb{Z}$.

Solution to Exercise 8.4. Expanding out both sides of $(e^{i\theta})^3 = e^{3i\theta}$ gives the following equalities.

$$\cos(3\theta) + i\sin(3\theta) = e^{3i\theta} = (e^{i\theta})^3 = (\cos \theta + i\sin \theta)^3$$
$$= \cos^3 \theta - 3\cos \theta \sin^2 \theta + i(3\cos^2 \theta \sin \theta - \sin^3 \theta).$$

The real part of this equality gives the desired result:

$$\cos(3\theta) = \cos^3 \theta - 3\cos \theta \sin^2 \theta.$$

Solution to Exercise 8.5.

$$\int \cos 2\theta \cos 3\theta \, d\theta = \int \frac{e^{2i\theta} + e^{-2i\theta}}{2} \cdot \frac{e^{3i\theta} + e^{-3i\theta}}{2} \, d\theta$$
$$= \int \frac{e^{5i\theta} + e^{-5i\theta} + e^{i\theta} + e^{-i\theta}}{4} \, d\theta = \frac{1}{2} \int (\cos 5\theta + \cos \theta) \, d\theta$$
$$= \frac{1}{2} \left( \frac{\sin 5\theta}{5} + \sin \theta \right) + C.$$

Solution to Exercise 8.6.

$$\int \sin^2 \theta \, d\theta = \int \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^2 \, d\theta = \int \frac{e^{2i\theta} + e^{-2i\theta} - 2}{-4} \, d\theta$$
$$= -\frac{1}{2} \int (\cos 2\theta - 1) \, d\theta = -\frac{1}{2} \left( \frac{\sin 2\theta}{2} - \theta \right) + C.$$
Solution to Exercise 8.7.
\[ \cos s \cos t = \frac{e^{is} + e^{-is}}{2} \cdot \frac{e^{it} + e^{-it}}{2} = \frac{e^{i(s+t)} + e^{-i(s+t)} + e^{i(s-t)} + e^{-i(s-t)}}{4} = \frac{\cos(s + t) + \cos(s - t)}{2}. \]

Solution to Exercise 9.1. \( y'' - 2y' + 10y = 0 \) has characteristic equation \( a^2 - 2a + 10 = 0 \) which has roots \( a = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i \). Thus \( e^{1+3i} \) is a complex solution and the general real solution is
\[ C_1 e^{t} \cos 3t + C_2 e^{t} \sin 3t. \]

Solution to Exercise 9.2. \( 2y'' + 2y' + y = 0 \) has characteristic equation \( 2a^2 + 2a + 1 \) which has roots \( a = \frac{-2 \pm \sqrt{4 - 8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i \). Thus \( e^{-\frac{1}{2}+\frac{1}{2}i} \) is a complex solution and the general real solution is
\[ C_1 e^{-\frac{1}{2}t} \cos \frac{t}{2} + C_2 e^{-\frac{1}{2}t} \sin \frac{t}{2}. \]

Solution to Exercise 9.3. We solve \( 9y'' + 6y' + 82y = 0 \) subject to the conditions \( y(0) = -1 \) and \( y'(0) = 2 \). The characteristic equation is \( 9a^2 + 6a + 82 = 0 \) which has roots \( a = \frac{-6 \pm \sqrt{36 - 4 \cdot 9 \cdot 82}}{18} = -\frac{1}{3} \pm \frac{3}{3}i \). Thus \( e^{-(1+3i)t} \) is a complex solution. One can then proceed to a general solution and solve for the coefficients. We’ll mimic our earlier arguments. It will help things to restate our initial conditions so that one of the conditions is homogeneous. Specifically, we will find \( c \in \mathbb{C} \) so that \( y(t) = \text{Re} (ce^{-(1+3i)t}) \) satisfies \( y(0) = -1 \) and \( y'(0) + 2y(0) = 0 \). We have
\[ \left( ce^{-(1+3i)t} \right)_{t=0} + 2 \left( ce^{-(1+3i)t} \right)_{t=0} = (-\frac{1}{3} + 3i) + 2 = \frac{5}{3} + 3i. \]

Then the real part of \( i(\frac{5}{3} - 3i) e^{-(1+3i)t} \) will satisfy the homogeneous initial condition. Since
\[ \left[ i(\frac{5}{3} - 3i) e^{-(1+3i)t} \right]_{t=0} = 3 + \frac{5}{3}i, \]
the real part of \( \frac{x}{2} i(\frac{5}{3} - 3i) e^{-(1+3i)t} \) will satisfy both initial conditions. To complete this work, we simplify to make the answer explicitly in the usual
form.

\[
\text{Re}\left(\frac{-1}{3}i(\frac{5}{3} - 3i)e^{-(\frac{1}{3} + 3i)t}\right) = \frac{-1}{3}e^{-\frac{1}{3}t}\text{Re}\left((3 + \frac{5}{3}i)(\cos 3t + i\sin 3t)\right)
\]

\[
= \frac{-1}{3}e^{-\frac{1}{3}t}\left(3\cos 3t - \frac{5}{3}\sin 3t\right) = e^{-\frac{1}{3}t}\left(-\cos 3t + \frac{5}{9}\sin 3t\right)
\]

Solution to Exercise 9.4. a) The associated homogeneous equation is

\[y'' - 2y' + 5y = 0.\]

The roots of the characteristic equation are

\[r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.\]

Thus, a complex solution to the given differential equation is

\[y = e^{(1+2i)x} = e^x(\cos 2x + i\sin 2x).\]

b) The real and imaginary parts of the complex solution are independent real solutions, so the general solution we are after is

\[y = C_1e^x\cos 2x + C_2e^x\sin 2x.\]

c) We guess \(y = a + bx\) and hope that we can choose the constants to make the equation work. We get:

\[y = a + bx\]
\[y' = b\]
\[y'' = 0\]

\[y'' - 2y' + 5y = 5a - 2b + 5bx.\]

We see that we should choose \(b = \frac{1}{5}\) and \(a = \frac{7}{25}\). Thus our particular solution is given by

\[y = \frac{7}{25} + \frac{1}{5}x.\]

d) The general solution to the inhomogeneous equation is the sum of the general solution to the homogeneous equation and a particular solution to the inhomogeneous equation. Thus the general solution we are looking for is

\[y = C_1e^x\cos 2x + C_2e^x\sin 2x + \frac{2x}{25} + \frac{1}{5}.\]

\[\square\]
\textit{Solution to Exercise 10.1.} Inversion of the formula at (67) gives
\[ a_0 = c_0, \quad a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}) \]
for all \( k > 0 \). \hfill \square

\textit{Solution to Exercise 10.2.} Before we begin, we remind ourselves that for every integer \( k \), the function \( e^{-ikx} \) has period dividing \( 2\pi \), so \( e^{-ik\pi} = e^{-ik(-\pi)} \). Now we proceed: If \( f(x) = x \) we have
\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-ikx} \, dx = \frac{1}{2\pi} \left[ \frac{xe^{-ikx}}{-ik} - \frac{e^{-ikx}}{(-ik)^2} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left[ \frac{ixe^{-ikx}}{k} + \frac{e^{-ikx}}{k^2} \right]_{-\pi}^{\pi} = \frac{ie^{-ik\pi}}{k} = \frac{i(-1)^k}{k}
\]
for \( k \neq 0 \) and \( c_0 = 0 \). Thus, using the results of Exercise 10.1, we get \( a_0 = 0 \)
\[
a_k = \pm 2i \left( \frac{1}{k} + \frac{1}{-k} \right) = 0,
\]
and
\[
b_k = i^2(-1)^k \left( \frac{1}{k} - \frac{1}{-k} \right) = (-1)^{k+1} \frac{2}{k}.
\]
We then can write our answer
\[
x \sim 2 \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - + \cdots \right).
\]
\hfill \square

\textit{Solution to Exercise 10.3.} In the Fourier series
\[
\frac{1}{2} - \frac{1}{3} \sin x + \frac{1}{4} \sin 2x - \frac{1}{5} \sin 3x - + \cdots
\]
we have \( a_0 = 0, \ a_k = 0, \) and \( b_k = (-1)^k \frac{1}{k+2} \) for \( k > 0 \). Equation (68) then gives \( c_0 = 0 \),
\[
c_k = \frac{1}{2} \left( 0 - i(-1)^k \frac{1}{k} \right) = i \frac{(-1)^{k+1}}{2(k+2)},
\]
and
\[
c_{-k} = \frac{i(-1)^k}{2(k+2)}
\]
for \( k > 0 \). We can then write our answer as
\[
\frac{i}{2} \left( \cdots + \frac{e^{-2ix}}{4} - \frac{e^{-ix}}{3} + \frac{e^{ix}}{3} - \frac{e^{2ix}}{4} + + \cdots \right).
\]
\hfill \square
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REFERENCES


