Thomas Hunter April 6, 2005

answers

Math 18 (2)
Second Midterm

Show your work. Correct answers with no justification may receive little or no credit. No calculators are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space.

Each part of each problem is worth 9 points. There are eleven parts altogether, in three problems.

Problem 1. In this problem we consider the region $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$ and the function $f : D \to \mathbb{R}$ defined by $f(x, y) = x^2y - y$.

a) (9 points) Find the critical points of f which are in the interior of D.

Solution. We have $\vec{\nabla} f = (2xy, x^2 - 1)$. In order for the second component of $\vec{\nabla} f$ to be zero, we must have $x = \pm 1$. Thus, in order for both components to be zero, we must also have $y = 0$. This gives the critical points $(1, 0)$ and $(-1, 0)$. Both of these points are in D. □

b) (9 points) Use a second derivative test to classify each of the critical points you found in part a as a local minimum, a local maximum, or a saddle point.

Solution. The matrix of second derivatives is $Hf = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix}$ which has determinant $\det Hf = -4x^2$. This determinant is negative at both of the critical points, so they are both saddles. □

c) (9 points) Use the method of Lagrange multipliers to find a system of equations whose solutions are the constrained critical points of f on ∂D.

Solution. The constraint giving the boundary is $g(x, y) = x^2 + y^2 = 4$. So the system we want is the two equations given by $\vec{\nabla} f = \lambda \vec{\nabla} g$ together with the constraint equation $g(x, y) = 4$. Writing these down we get

$$2xy = \lambda 2x \quad (1)$$
$$x^2 - 1 = \lambda 2y \quad (2)$$
$$x^2 + y^2 = 4 \quad (3)$$

□

d) (9 points) Solve the system of equations you found in part c.

Solution. From equation (1) we see that either $x = 0$ or $y = \lambda$. We consider these two cases separately. If $x = 0$ equation (3) becomes $y^2 = 4$ so $y = \pm 2$–this gives constrained extrema at $(0, \pm 2)$. If $y = \lambda$ equations (2) and (3) become

$$\begin{aligned} x^2 - 2y^2 &= 1 \\ x^2 + y^2 &= 4. \end{aligned}$$

This is a linear system in x^2 and y^2 which is easy to solve giving $x^2 = 3$ and $y^2 = 1$. This gives four more constrained critical points $(\pm\sqrt{3}, \pm 1)$. constrained extrema. □

e) (9 points) Use your work in parts a through d to find the absolute maximum and absolute minimum of f on D.

Solution. Here we just evaluate our function at all of the critical points–both constrained and otherwise. We leave out the saddles, since we know they are not extreme.

$$\begin{aligned} f(0, -2) &= 2 \\ f(0, 2) &= -2 \\ f(\sqrt{3}, 1) &= 2 \\ f(-\sqrt{3}, 1) &= 2 \\ f(\sqrt{3}, -1) &= -2 \\ f(-\sqrt{3}, -1) &= -2. \end{aligned}$$

□

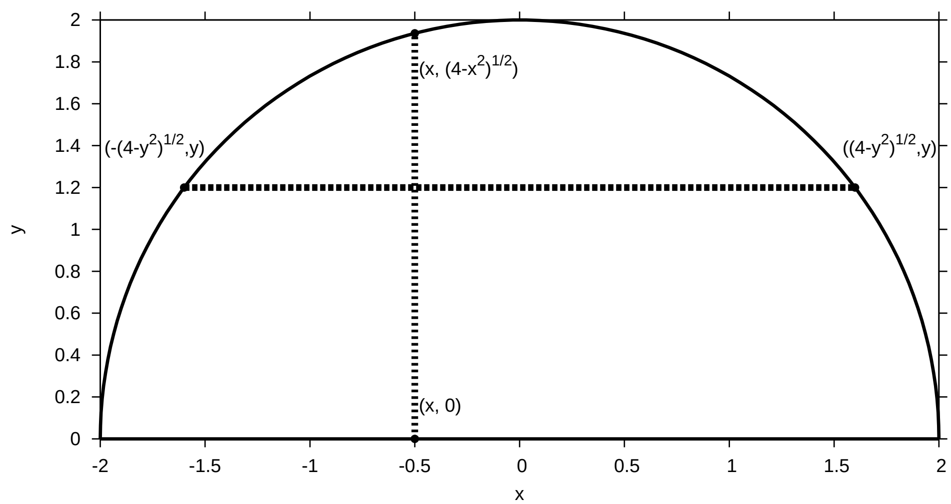
We see that the maximum is $f(0, -2) = f(\pm\sqrt{3}, 1) = 2$ and the minimum is $f(0, 2) = f(\pm\sqrt{3}, -1) = -2$.

Problem 2. In this problem we consider the integral

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{x^2+y^2}\, dx\, dy.$$

a) (9 points) Draw the region of integration.

Solution. The region is the half-circle pictured below. The original order of integration corresponds to the slices drawn horizontally in the picture.



□

b) (9 points) Give an equivalent integral with the order of integration reversed.

Solution. For this, we want to imagine the region sliced vertically. Reading of of our picture, we get

$$\int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx.$$

□

c) (9 points) Give an equivalent integral in polar coordinates.

Solution.

$$\int_{\theta=0}^{\pi}\int_{r=0}^{2} e^{r^2} r\,dr\,d\theta$$

□

d) (9 points) Compute the integral using whichever form (the original, the one from part b, or the one from part c) makes your work the easiest.

Solution.

$$\int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx = \pi\int_0^2 \frac{1}{2} d\left(e^{r^2}\right) = \frac{\pi}{2}\left[e^{r^2}\right]_0^2 = \frac{\pi}{2}(e^4-1).$$

□

Problem 3. In this problem, let $\vec{F}:\mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\vec{F}(x,y) = (y+1, x+1).$$

Also, let C be the circle with center $(1,0)$ and radius 2 oriented counterclockwise.

a) (9 points) Compute $\oint_C \vec{F}\cdot d\vec{s}$ by giving an explicit parametrization of C and applying the definition of vector line integral.

Solution. We parametrize C as follows:

$$\begin{aligned} x &= 2\cos t + 1 \qquad & y &= 2\sin t \\ dx &= -2\sin t\,dt \qquad & dy &= 2\cos t\,dt \end{aligned}$$

where $0 \le t \le 2\pi$. We then substitute into the integral to get

$$\oint_C \vec{F}\cdot d\vec{s} = \int_C (y+1)dx + (x+1)dy = \int_0^{2\pi} ((2\sin t + 1)(-2\sin t) + (2\cos t + 2)(2\cos t))\,dt$$
$$= \int_0^{2\pi} (4\cos(2t) + 4\cos t - 2\sin t)\,dt = [2\sin(2t) + 4\sin(t) + 2\cos t]_0^{2\pi} = 0.$$

□

b) (9 points) Explain how you could have obtained your answer to part a *without* any nontrivial integration.

Solution. Since the circle, C, is the boundary, ∂D, of a disk, D, on which $\vec{F}$ is defined, we have by Green's theorem

$$\oint_C \vec{F}\cdot d\vec{s} = \iint_D \vec{\nabla}\times\vec{F}\cdot\vec{k}\,dV = \iint_D \left(\frac{\partial}{\partial x}(x+1) - \frac{\partial}{\partial y}(y+1)\right) dx\,dy = \iint_D (1-1)dx\,dy = 0.$$

□