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Math 103
Final Exam

Write all of your work in your bluebooks. You may write the problems in any order you like, but do not put work for more than one problem on the same page of your bluebook. When you are done, number the pages of your bluebook(s) and make a table of contents on the cover of the first one indicating which problems you worked and which pages I can find them on.

Each problem is worth 15 points.

This exam is closed book and closed notes. However you may ask me questions or ask for hints. In particular in the last problem, for a small penalty you may ask me the name of the theorem you should be citing. For a more substantial penalty, you can ask for a statement of that theorem.

Problem 1. What are the Cauchy-Riemann equations and why do they hold for a complex analytic function?

Problem 2. Let a, b be complex numbers with $a \neq b$. Give an explicit branch of $\log\left(\frac{z-a}{z-b}\right)$.

Problem 3. Find all Laurent expansions centered at i of the function $f(z) = \frac{1}{z^2-z}$. For each expansion, indicate the largest open set on which it converges.

Problem 4. Use the theory of residues to calculate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Problem 5. Find a conformal mapping of the half-disk, $B = \{|z| < 1, \mathrm{Im}(z) > 0\}$ onto the disk $\mathbb{D} = \{|z| < 1\}$. Draw a picture indicating the images under your mapping of the arcs of the circles in B which pass through 1 and -1.

Problem 6. Prove the following theorem. In doing so, you may cite other, weaker, theorems from our text as long as you state them clearly—including all necessary hypotheses.

Theorem. *Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = N_0 - N_\infty$, where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.*

Problem 7. Let $f(z)$ be defined by the formula

$$f(z) = \begin{cases} 1 & \text{if } \mathrm{Im}(z) \geq 0, \\ 0 & \text{if } \mathrm{Im}(z) < 0. \end{cases}$$

For each of the following statements either give a careful proof or a careful disproof. (You may cite theorems from our text without proof as long as you state them clearly—including all necessary hypotheses.)

1) There is a sequence of polynomials $\{p_n(z)\}_{n \geq 0}$ which converge normally to f on $\mathbb{C}$.

2) There is a sequence of polynomials $\{p_n(z)\}_{n \geq 0}$ such that on every compact subset, $K \subseteq \mathbb{C} - \mathbb{R}$, p_n converges uniformly to f on K.