ON THE WEIERSTRASS PRODUCT THEOREM

This short note is in response to some confusion about Gamelin's proof of the Weierstrass product theorem given in [Gam01, page 358]. Of course, the theorem is correct as stated. However, I do not see how the poles of the g_k used in the proof can be made to reside at w_k alone. In addition, the treatment given below makes it clear that we don't need to cite any general theorem about simple connectivity.

First, for reference, we will repeat the statement (but not the proof) of the sharper version of Runge's theorem, [Gam01, page 344].

Theorem 1. *Let* K *be a compact subset of* $\mathbb{C}$*,* f *be analytic on an open set containing* K*,* S *be a subset of* $\mathbb{C}^*$ *such that each component of* $\mathbb{C}^* - K$ *contains an element of* S*, and* $\epsilon > 0$*. Then there is a rational function* g *with poles in* S *such that* $|(f-g)(z)| < \epsilon$ *for every* z *in* K*.*

Next, we will state a simple lemma. You should be able to supply a very explicit formula for the asserted branch, using your understanding of the logarithm function, and without appealing to any theorems about simple connectivity. In this lemma and below, we will use the convention that when $w_0 = \infty$, the function $f(z) := \frac{z-z_0}{z-w_0}$ is given by $f(z) := z - z_0$.

Lemma 2. *Let* z_0 *be in* $\mathbb{C}$*,* w_0 *be in* $\mathbb{C}^*$*, and* γ *be the line segment,* $[z_0, w_0]$*, from* z_0 *to* w_0*, including the ends. Then an analytic branch of* $\log \frac{z-z_0}{z-w_0}$ *exists on* $\mathbb{C} - \gamma$*.*

Next we state a proposition which I wrote on the board in seminar last week.

Proposition 3. *Let* D *be a domain and define*

$$K_m := \left\{ z \in D \cup \partial D \mid |z| < m,\ \forall d \in \partial D. |z - d| \geq \frac{1}{m} \right\}.$$

Then the K_m *have the following properties:*

(1) $K_0 = \emptyset$.
(2) *Each* K_m *is a compact subset of* D.
(3) *For all* m*,* $K_m \subseteq K_{m+1}$.
(4) $\bigcup_m K_m = D$.
(5) *For all* m *and for any* z *outside of* K_m*, there is a* w *in* $\partial D \cup \{\infty\}$ *so that the interval* $[z, w]$ *is entirely outside of* K_m.
(6) *For every compact* $L \subseteq D$ *there is some* m *such that* $L \subseteq K_m$.

We are now ready to state and prove the Weierstrass product theorem. Our proof is basically the same as Gamelin, but with special attention to the points mentioned at the beginning of this note.

Theorem 4. *Let* $D \subseteq \mathbb{C}$ *be a domain,* $\{z_k\}_{k=1}^{\infty}$ *be a sequence of distinct points in* D *with no accumulation point in* D*, and* $\{n_k\}_{k=1}^{\infty}$ *be a sequence of integers. Then there is a meromorphic function* f *on* D *whose only zeros and poles occur at the* z_k*, such that the order of* f *at* z_k *is* n_k*.*

Proof. For each integer $m \geq 0$ let K_m be defined as in Proposition 3. For each integer $k \geq 1$, let m_k be the unique integer such that z_k is in $K_{m_k+1} - K_{m_k}$. For each k, let $w_k \in \partial D \cup \infty$ be the element provided by part 5 of Proposition 3, where z in the Proposition is taken to be z_k and m is taken to be m_k. Thus $[z_k, w_k]$ is entirely outside of K_{m_k}. Now apply Lemma 2 get a branch, $f_k(z)$, of $\log \frac{z-z_k}{z-w_k}$ which is analytic on $\mathbb{C} - [z_k, w_k]$. Since $[z_k, w_k]$ is entirely outside of K_{m_k}, f_k is analytic on an open set containing K_{m_k}. We now apply Theorem 1, where K in the Theorem is taken to be K_{m_k}, f is taken to be $n_k f_k$, S is taken to be $\partial D \cup \{\infty\}$, and ϵ is taken to be $\frac{1}{2^k}$. Write g_k for the function which Theorem 1 provides for us in this context. Thus we have $|f_k - g_k| < \frac{1}{2^k}$ on K_{m_k}. Thus, the Weierstrass M-test tells us that $\sum_{k=1}^{\infty}(f_k - g_k)$ converges uniformly on K_m for every m, and normally on D in light of part 6 of Proposition 3. Thus, by the definition of convergence for infinite products, the infinite product $\prod_{k=1}^{\infty} e^{f_k(z)-g_k(z)}$ converges normally on D. Call the limit of this infinite product $f(z)$. We claim that $f(z)$ has the required properties. In fact, by definition, we have

$$f(z) = \prod_{k=1}^{\infty} \left(\frac{z - z_k}{w - w_k} \right)^{n_k} e^{-g_{k(z)}}.$$

The function e^{-g_k} is entire and has no zeros. $\frac{1}{z-w_k}$ is analytic in D and has no zeros. Thus f has zeros and poles only at the points z_k and the order of f at z_k is n_k.

□

References

[Gam01] Theodore W. Gamelin. *Complex analysis.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.