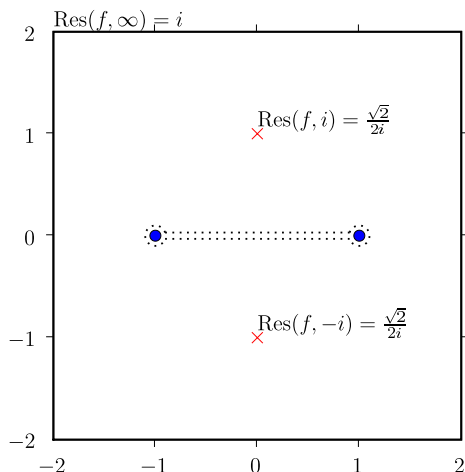


Proposition.

$$\int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx = \pi(\sqrt{2} + 1).$$

Here is a figure to which we will refer during our discussion.



Proof. Let $f(z) = g(z)/(1+z^2)$ where $g(z)$ is defined by the following formula in which we mean the principal square root each time we write $\sqrt{\cdot}$.

$$g(z) = \begin{cases} \sqrt{1-z^2} & \text{if } \text{Im}(z) > 0, \\ -i\sqrt{z^2-1} & \text{if } z \in (1, \infty), \\ i\sqrt{z^2-1} & \text{if } z \in (-\infty, 1), \\ -\sqrt{1-z^2} & \text{if } \text{Im}(z) < 0. \end{cases}$$

The reader may check¹ that this defines a continuous function on $D = \mathbb{C} - [-1, 1]$. Morerra's theorem then shows that $f(z)$ is analytic on D . We consider the integral of $f(z)$ along a

¹Well, I wrote down a discontinuous function in the first version of these notes, so I guess I should give the argument that $g(z)$ is, in fact, continuous on D . The potential discontinuities of g are those z for which $1-z^2$ is a point of discontinuity for the principal square root. That is to say those real z with $|z| > 1$. On $(1, \infty)$ what we need to check is that if $a > 1$, and $\epsilon > 0$ is then the differences between the three numbers

$$\sqrt{1-(a+\epsilon i)^2}, \quad -i\sqrt{a^2-1}, \quad -\sqrt{1-(a-\epsilon i)^2}$$

go to zero with ϵ . In fact we have

$$\begin{aligned} \sqrt{1-(a+\epsilon i)^2} &= \sqrt{1-a^2+\epsilon^2-2a\epsilon i} = -i\sqrt{a^2-1} + \mathcal{O}(\epsilon) \\ -\sqrt{1-(a-\epsilon i)^2} &= -\sqrt{1-a^2+\epsilon^2+2a\epsilon i} = -\left(i\sqrt{a^2-1} + \mathcal{O}(\epsilon)\right) \end{aligned}$$

so g is continuous at the points in $(1, \infty)$. The argument for continuity along $(-\infty, 1)$ is similar.

dumbbell-shaped path Γ_ϵ such as the one drawn in the picture. We will assume that the radius of the circular parts of Γ_ϵ goes to zero with ϵ and we will make a similar assumption about the separation of the linear segments. We will take the clockwise orientation of Γ_ϵ and consider it to be the boundary of an external domain.

Since $f(z)$ is bounded in neighborhoods of 1 and -1 , and since the length of the circular parts of Γ_ϵ go to zero with ϵ , we see that the contribution to $\int_{\Gamma_\epsilon} f(z) dz$ from these circular sections goes to zero with ϵ . If x is in $[-1, 1]$, then as z approaches x through values in the upper half plane $f(z)$ approaches $\frac{\sqrt{1-x^2}}{1+x^2}$. Similarly, as z approaches such an x through values in the lower half plane $f(z)$ approaches $-\frac{\sqrt{1-x^2}}{1+x^2}$. It follows that

$$\int_{\Gamma_\epsilon} f(z) dz \rightarrow 2 \int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx \text{ as } \epsilon \rightarrow 0.$$

On the other hand, the residue theorem tells us that $\int_{\Gamma_\epsilon} f(z) dz$ is $2\pi i$ times the sum of the residues of f in the exterior domain determined by Γ_ϵ including the residue at ∞ . The poles of f are simple poles at $\pm i$. An easy calculation from the formula for $f(z)$ shows that $\text{Res}(f, i) = \text{Res}(f, -i) = \frac{\sqrt{2}}{2i}$. We will show below that $\text{Res}(f, \infty) = -i$ and then we will have

$$\int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx = \pi i \left[\frac{\sqrt{2}}{i} + i \right] = \pi(\sqrt{2} - 1)$$

as desired. It remains to calculate the residue at infinity. We have

$$\begin{aligned} f(z) &= \pm \frac{\sqrt{1-z^2}}{1+z^2} = \pm \frac{i}{z} (1-z^{-2})^{\frac{1}{2}} (1+z^{-2})^{-1} = \pm \frac{i}{z} \left(\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} z^{-2k} \right) \left(\sum_{k=0}^{\infty} \binom{-1}{k} z^{-2k} \right) \\ &= \pm \frac{i}{z} (1 + \mathcal{O}(z^{-2}))(1 + \mathcal{O}(z^{-2})) \text{ as } z \rightarrow \infty. \end{aligned}$$

Since the original formula for $f(z)$ shows us that $f(z)$ is pure imaginary and in the lower half plane when z is real and bigger than 1 , we see we must take \pm to be $-$ in the above series expansion for $f(z)$. Thus $\text{Res}(f, \infty) = i$ as desired. \square