Proposition. *Let* z_0 *an isolated singularity of the analytic function* f. *Suppose that* m *is an integer such that* $f = \mathcal{O}((z - z_0)^m)$ *in the sense that there is a constant* C *such that* $|f(z)| \leq C|(z - z_0)^m|$ *for* z *near* z_0. *Let* $\sum_k a_k(z - z_0)^k$ *be the Laurant series expansion for* f *in a punctured disk centered at* z_0. *Then for each* $k < m$ *we have* $a_k = 0$. *That is to say* $f = \mathcal{O}((z - z_0)^m)$ *in the sense that*

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots .$$

Proof. Let k be an integer less than m. Then for each small ϵ the ML estimate gives us

$$|a_k| = \left| \frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} \frac{f(z)}{(z - z_0)^{k+1}} dz \right| \leq \frac{1}{2\pi} \frac{C\epsilon^m}{e^{k+1}} 2\pi\epsilon = C\epsilon^{m-k}.$$

Thus $a_k = 0$ as desired. □

Remark. The reader should be able to provide the easy proof of the converse.