Given two sets disjoint $M, W$ of $n$ elements each, a matching is an assignment of each $m \in M$ to a distinct $w \in W$ (e.g., a one-to-one function).

Suppose each element in each set has a preference order for all the elements in the other set, that is, a ranking of all $n$ of those elements, with no ties. Then the matching is stable if there are no pairs $(m, w)$ and $(m', w')$ in the matching such that $m$ prefers $w'$ to $w$ and $w'$ prefers $m$ to $m'$. If such a duet of pairs does exist, it is called an instability in the matching.

**Theorem 1.** Given $n$-sets $M, W$ with any preference orders, there always exists a stable matching.

Proof: Uses the matching algorithm discussed in class.

Preferences between whole matchings:

Given two matchings $\mathcal{M}$ and $\mathcal{M}'$ of $M$ to $W$, we say that set $M$ prefers $\mathcal{M}$, and write $\mathcal{M} >_m \mathcal{M}'$, if every $m \in M$ is at least as well off in $\mathcal{M}$ as in $\mathcal{M}'$. Similarly one defines $>_w$.

**Theorem 2.** For any stable matchings $\mathcal{M}, \mathcal{M}'$, we have $\mathcal{M} >_m \mathcal{M}' \iff \mathcal{M} <_w \mathcal{M}'$.

**Theorem 3.** The matching $\mathcal{M}^*$ created by the Algorithm when set $M$ chooses is preferred by $M$ to all other stable matchings (if there are any). That is, $\mathcal{M}^*$ is optimal for set $M$ in the collection of all stable matchings.

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