The problem:

A couple have a party with $n$ other couples. At the party, each person shakes hands once with each person s/he did not already know, and no other people. (Each couple know each other so they don’t shake hands.) At the end of the party, the hostess asks everyone else (including her partner) how many times they shook hands. Surprisingly, all the numbers are different. How many times did the hostess shake hands? How many times did her partner shake hands?

Solution:

First some definitions. Call the conditions in the problem on a party with a host couple and $n$ guest couples the **Halmos conditions of order $n$**.

Next, for any party let the **party graph** be the graph with one vertex for each person at the party and an edge between two vertices if the people they represent shake hands at the party.

Finally, define a graph $G_n$ as follows. There are $2(n+1)$ vertices arranged in $n+1$ vertical pairs from left to right, e.g., for $n=3$

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\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Furthermore, the complete set of edges is obtained this way: for each bottom vertex, draw an edge to each vertex to its left, on either bottom or top. E.g., for $n=2$ we have

Another way to think about $G_n$ is: each bottom vertex is adjacent to every other bottom vertex and to every top vertex to its left. Each top vertex is adjacent just to bottom vertices to its right. The degrees of the top vertices, from left to right, are $n$, $n-1$, $n-2$, $\ldots$, $2$, $1$, $0$. The degrees of the bottom vertices are, from left to right, $n$, $n+1$, $n+2$, $\ldots$, $2n-1$, $2n$. (Remember, there are $2n+2$ vertices total.)

**Theorem.** If a party with $n$ guest couples and a host couple meets the Halmos conditions, then the party graph must be $G_n$, where each vertical pair of vertices represents a couple and the host couple is the leftmost pair. In particular, both the hostess and host shake hands exactly $n$ times.

Proof: For each $n$, let $P(n)$ be the claim in the Theorem for that $n$. We prove $P(n)$ for all integers $n \geq 0$ by induction.
Basis $P(0)$. In this case the “party” consists of the host and hostess. By the Halmos conditions they don’t share hands, and sure enough, the other condition, that each of the 1 partygoers other than the hostess shakes hands a different number of times, is vacuously true. Also sure enough, the conclusions are true: the party graph is $G_0$ (two nonadjacent vertices), the host pair is the leftmost (in fact only) pair, and the hostess and host both shake hands $n = 0$ times.

Inductive step, $P(n-1) \implies P(n)$. We assume the theorem for parties with $n-1$ guest couples and are asked to prove it when there are $n$ guest couples. So assume we are presented with a party with $n$ guest couples that meets the Halmos conditions. The most handshakes anyone can do is $2n$ (everyone in the $n$ other couples). Since every one of the $2n + 1$ people other than the hostess shakes hands a different number of times, each of the numbers from 0 to $2n$ must be the number of handshakes for one of them.

So consider the person among them who shakes hands $2n$ times. This person shakes hand with everyone other than his/her partner. Therefore, everyone except this partner shakes hands at least once, so this partner must be the nonhostess person who shakes hands 0 times. Therefore, representing this couple as the rightmost pair is consistent with the party graph being $G_n$.

In any event, temporarily delete this couple from the party. Call the result the reduced party. We claim that the reduced party meets the Halmos conditions of order $n - 1$. (This must be verified in order to invoke $P(n-1)$, or else we are guilty of the same faulty induction as in the tournament cyclic ordering problem.) First, the host and hostess are still in the party. Second, if the hostess is excluded, everyone shakes hands a different number of times (because they all shook hands a different number of times in the original party, and they now each shake hands exactly one less time than before). Thus the reduced party meets the Halmos conditions of order $n - 1$.

Thus, by $P(n-1)$, the reduced party may be represented as $G_{n-1}$, with the host couple on the far left and each vertical pair a couple. Now put back the deleted couple on the far right, along with their edges, and we have $G_n$, with the host couple still on the far left and with each couple still a vertical pair. ■