Special Method to Find Polynomial Formulas from Difference Tables

I referred to this twice in class, but each time I stopped in the middle because I could see that there was something wrong with my explanation – I wasn’t going to get the right formula. So I owe you the right explanation and some examples. Here they are.

The reason my formula wasn’t working is that it only works if you start your sequence with the zeroth term, \( a_0 \), not the first term, \( a_1 \). A slightly different formula works when you start with the first term.

So let the sequence be \( a_0, a_1, \ldots, a_n \), and make a table of differences until you get a row of zeros (which will happen if and only if there is a polynomial formula for \( a_n \)). Let the first entry in the top row be called \( f_0 \) (it is also \( a_0 \)), the first entry in the next row \( f_1 \), and so on. For instance, consider

\[
\begin{array}{cccccc}
1 & 1 & 7 & 25 & 61 & 121 & 211 \\
0 & 6 & 18 & 36 & 60 & 90 \\
6 & 12 & 18 & 24 & 30 \\
6 & 6 & 6 & 6 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Then the top row is \( a_0, a_1, a_2, a_4, a_5, a_6 \), and \( f_0 = 1 \), \( f_1 = 0 \), \( f_2 = f_3 = 6 \), and \( f_4 = 0 \).

Now, the formula for \( a_n \) is

\[
a_n = f_0 + f_1 n + \frac{f_2}{2} n(n - 1) + \frac{f_3}{6} n(n - 1)(n - 2).
\]

If there were another nonzero row, the next term would be

\[
\frac{f_4}{24} n(n - 1)(n - 2)(n - 3),
\]

and so on. In general, it is a theorem that if the last nonzero row is row \( k \), then the formula is

\[
a_n = \sum_{j=0}^{k} \frac{f_j}{j!} \prod_{i=0}^{j-1} (n - i).
\]  (1)

For instance, for the sequence 1,1,7,25,61,\ldots above, we compute

\[
a_n = 1 + 0n + 3n(n - 1) + 1n(n - 1)(n - 2)
\]

\[
= 1 + (3n^2 - 3n) + (n^3 - 3n^2 + 2n)
\]

\[
= n^3 - n + 1.
\]

You can check that this is right. For instance, \( a_0 = 0^3 - 0 + 1 = 1 \), and \( a_1 = 1^3 - 1 + 1 = 1 \).
If a sequence starts with index 1, that is, we have \( a_1, a_2, a_3, \ldots, a_n \), then there is merely a “change by 1” in the terms of the formula
\[
a_n = f_0 + f_1(n-1) + \frac{f_2}{2} (n-1)(n-2) + \frac{f_3}{6} (n-1)(n-2)(n-3) + \cdots
\]
\[
= \sum_{j=0}^{k} \frac{f_j}{j!} \prod_{i=1}^{j} (n-i)
\]
(2)

For instance, let’s lop the first term off the original sequence, and thus off all the rows below it in the first display, obtaining

\[
\begin{array}{ccccccc}
1 & 7 & 25 & 61 & 121 & 211 \\
6 & 18 & 36 & 60 & 90 \\
12 & 18 & 24 & 30 \\
6 & 6 & 6 \\
0 & 0
\end{array}
\]

Then the revised formula tells us that
\[
a_n = 1 + 6(n-1) + 6(n-1)(n-2) + 1(n-1)(n-2)(n-3)
\]
\[
= 1 + (6n-6) + (6n^2 - 18n + 12) + (n^3 - 6n^2 + 11n - 6)
\]
\[
= n^3 - n + 1.
\]

So sure enough, the final formula is the same (but we apply it starting at \( n = 1 \) not \( n = 0 \)).

**Exercise.** Apply this method to the sequence Delia analyzed:

\[
0, 0, 6, 24, 60, 120, 210, 336, 504, 720.
\]

Remember, the first term is \( a_1 \), so you should use the second method. The final answer should be \( a_n = n(n-1)(n-2) \).

**Curious why this works?** It’s in DAM, but only as a problem late in the book, [13. Ch 9 Supplementary Problems]. If you are willing to read Sect 9.4 (which is almost independent of earlier material) you should be able to do this problem, thus justifying the method.