Power Series and: Indeterminates, Probability

1. Find

a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{1-\cos x}$

b) $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{1-\cos x}$ Note that x now approaches 0 from the positive side only.

c) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$

2. I have denigrated L'Hoptial's Rule as a way to compute indeterminate forms. I've been a bit unfair; for expressions of the form $f(x)/g(x)$ where x goes to *infinity*, L'Hopital's Rule still applies, but the power series method usually doesn't work. (Why? Because you can't usually write a power series with $a = \infty$.) However, sometimes power series can still be made to work.

You probably know that, for any power n, $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$. This is easy to prove with L'Hopital's Rule. Here is a power series proof:

Step 1: for all $x > 0$, $\displaystyle e^x > \frac{x^{n+1}}{(n+1)!}$ Why?

Step 2: $\displaystyle\lim_{x\to+\infty} \frac{x^{n+1}/(n+1)!}{x^n} = +\infty.$ Why?

Step 3: $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty.$ Why?

3. As in class, let p_n = the probability that you get your first 6 on the nth toss of a fair die. We found that $p_n = \left(\frac{5}{6}\right)^{n-1}\frac{1}{6}$. Further, we defined $\displaystyle P(x) = \sum_{n=1}^{\infty} p_n x^n$ and then found that $\displaystyle P(x) = \frac{x}{6-5x}$.

a) Let q_n = the probability that you *don't* get your first 6 on the nth throw, that is, you either don't get a 6 at all or you don't get your first 6. (I assume you keep tossing forever.) Show that $\displaystyle\sum_{n=1}^{\infty} q_n x^n = \frac{x}{1-x} - \frac{x}{6-5x}$. (This is easy to see if you think about how the numbers p_n and q_n are related.)

b) In answer to Patrick's question, I said that $P(2)$ equalled how much you should pay to play this toss game if you get \$$2^n$ when your first 6 is on the nth toss. However, show that what I said can't be right because $\displaystyle\sum_{n-=1}^{\infty} p_n x^n$ doesn't converge for $x = 2$. (The expression $x/(6-5x)$ makes fine sense for $x = 2$, but it doesn't equal the power series for $x = 2$.) What is the interval of convergence for $P(x)$?

c) Suppose your toss a tetrahedral die instead (4 sides) and you let r_n = the probability of getting your first 4 on the nth toss. Find the closed form for $R(x) = \sum_{n=1}^{\infty} r_n x^n$.