Complex Numbers, Indeterminate Forms, and Power Series

1. **a)** Find the magnitude and angle of $1+i\sqrt{3}$.

b) Predict without doing complex multiplication the magnitude and angle of $(1+i\sqrt{3})^2$.

c) Verify that you were correct by multiplying out $(1+i\sqrt{3})^2$.

2. Let $z = e^{a+bi}$, where a and b are real numbers. Explain why the magnitude of z is e^a and the angle is b. Hint: $e^{p+q} = e^p e^q$. Conclusion: both exponential and periodic behavior are aspects of the same e^x function, depending on whether you look at the real or imaginary part of x.

3. Using the facts that

$$\begin{aligned} e^{i\alpha} &= \cos\alpha + i\sin\alpha, \\ e^{i(-\beta)} &= \cos(-\beta) + i\sin(-\beta) = \cos\beta - i\sin\beta, \end{aligned}$$

it follows that

$$e^{i\alpha}e^{i(-\beta)} = (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta).$$

Simplify the left-hand side, and expand the right-hand side to prove the usual formulas for $\cos(\alpha-\beta)$ and $\sin(\alpha-\beta)$.

4. Derive the same formulas as in Problem 3 by dividing $e^{i\alpha} = \cos\alpha + i\sin\alpha$ by $e^{i\beta} = \cos\beta + i\sin\beta$ and simplifying. Do this problem only if you already know how to write a fraction $\dfrac{a+bi}{c+di}$ in the form $p+qi$ by multiplying top and bottom by $c-di$.

5. In class I showed that $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$. Come up with a similar formula for $\sin\theta$.

6. I claimed in class that every trig identity could be proved by rewriting it in terms of powers of $e^{i\theta}$ and simplifying using laws of exponents. Well, here is an identity you've probably seen:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Prove it the way I claimed can be done.

7. There are formulas for $\cos 3\theta$ and $\sin 3\theta$ that you probably haven't seen. Simultaneously find and prove them starting with

$$(\cos\theta + i\sin\theta)^3 = \left(e^{i\theta}\right)^3 = e^{3i\theta} = e^{i(3\theta)} = \cos 3\theta + i\sin 3\theta.$$

8. Find the Taylor Series around $a=0$ for

a) $\cos ix$ **b)** $\sin ix$ **c)** $\cosh ix$

Any observations?

9. Use power series to find

a) $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}$ **b)** $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x(e^x-x-1)}$

Those of you who know and like L'Hopital's Rule might try doing b) using that method.

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