More Power Series Manipulation

More Long Division

1. Find $\dfrac{1}{1+2x+3x^2+4x^3+\cdots}$ by very long division. Miraculously, in this case you can get the *complete* answer, not just the first few terms of the quotient. Try it and see.

1A. Find the 3rd degree Taylor Polynomial for $1/(1-\sin x)$ around $a=0$ by long division.

Composition and Inversion

2. Get the 4th degree Taylor Polynomial for $e^{\sin x}$ around 0 by plugging the power series for $\sin x$ around 0 in for x in the power series around 0 for e^x, and regrouping.

Answer: $1+x+x^2/2+0x^3-x^4/8$

3. **a**) Check that the power series for e^x around $a=1$ is $e\sum_{k=0}^{\infty}\frac{(x-1)^k}{k!}$

b) Get the 4th degree Taylor Polynomial for $e^{\cos x}$ around 0 by plugging the power series for $\cos x$ around 0 in for x in the power series *around* 1 for e^x.

Answer: $e-\dfrac{e}{2}x^2+\dfrac{e}{6}x^4$.

4. Try to get the 4th degree Taylor Polynomial for $e^{\cos x}$ by plugging the power series for $\cos x$ around 0 in for x in the power series *around* 0 for e^x. Why is this harder than the approach in the previous problem?

Moral: To get the power series for $f(g(x))$ around $x=a$, plug the power series for $g(x)$ around $x=a$ into the power series for $f(u)$ around $u=g(a)$.

5. Suppose you didn't already know the power series for $\ln x$ around $x=1$. Call it $\sum_{k=0}^{\infty}a_k(x-1)^k$. Find a_0,a_1,a_2,a_3 by composing the unknown power series with the known power series for e^x. Namely,

$$x=\ln(e^x)=\sum_{k=0}^{\infty}a_k(e^x-1)^k=\sum_{k=0}^{\infty}a_k\left(\sum_{j=1}^{\infty}\frac{x^j}{j!}\right)^k. \tag{1}$$

(Note that j starts at 1 in the last sum; why?) The leftmost and rightmost expressions in (1) are power series, so they must be identical. Thus

$$\begin{aligned} x=0+1x+0x^2+\cdots \;=\; & a_0+a_1\big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\big)+ \\ & a_2\big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\big)^2+a_3\big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\big)^3+\cdots \end{aligned}$$

From this last display, solve for a_0,a_1,a_2,a_3. You can check your work because you do know the power series for $\ln x$ around $x=1$, or can figure it out other ways.