(Parts of) The Proof of the Error Theorem

Suppose $f'''(x) \le M$ for all x in $[0,d]$. Integrating once we get: For any $u \in [0,d]$

$$\begin{aligned} \int_0^u f'''(t)\,dt \; &\le \; \int_0^u M\,dt \qquad (1) \\ f''(t)\Big|_0^u \; &\le \; Mt\Big|_0^u \\ f''(u)-f''(0) \; &\le \; Mu. \end{aligned}$$

Integrating again, remembering that $f''(0)$ is just a constant, we get: For any $v \in [0,d]$,

$$\begin{aligned} \int_0^v f''(u)-f''(0)\,du \; &\le \; \int_0^v Mu\,du \\ \int_0^v f''(u)\,du - \int_0^v f''(0)\,du \; = \; f'(u)\Big|_0^v - f''(0)\,u\Big|_0^v \; &\le \; \tfrac{1}{2}Mu^2\Big|_0^v \\ f'(v)-f'(0)-f''(0)v \; &\le \; \tfrac{1}{2}Mv^2. \end{aligned}$$

Integrating a third time, remembering that $f'(0)$ is also just a constant, we get: For any $x \in [0,d]$,

$$\begin{aligned} \int_0^x f'(v)-f'(0)-f''(0)v\,dv \; &\le \; \int_0^x \tfrac{1}{2}Mv^2\,dv \\ f(v)\Big|_0^x - f'(0)\,v\Big|_0^x - \tfrac{1}{2}f''(0)\,v^2\Big|_0^x \; &\le \; \tfrac{1}{3!}Mv^3\Big|_0^x \\ f(x)-f(0)-f'(0)x-\tfrac{1}{2}f''(0)\,x^2 \; &\le \; \tfrac{1}{3!}Mx^3 \qquad (2) \\ f(x)-P_2(x) \; &\le \; \tfrac{1}{3!}Mx^3. \end{aligned}$$

Had we started instead with basepoint a instead of 0, and integrated from a to d, the result would have been.

$$f(x)-f(a)-f'(a)(x-a)-\tfrac{1}{2}f''(a)\,(x-a)^2 \; \le \; \tfrac{1}{3!}M(x-a)^3.$$

Had we started with $-M \le f'''(x) \le M$, that is, with $|f'''(x)| \le M$, the triple integration would give us

$$-\tfrac{1}{3!}M(x-a)^3 \; \le \; f(x)-f(a)-f'(a)(x-a)-\tfrac{1}{2}f''(a)\,(x-a)^2 \; \le \; \tfrac{1}{3!}M(x-a)^3.$$

that is,

$$\big|f(x)-P_2(x)\big| \; \le \; \tfrac{1}{3!}M(x-a)^3.$$

Had we started with $-M \le f^{(n+1)}(x) \le M$ and integrated $n+1$ times, the result would be

$$\big|f(x)-P_n(x)\big| \; \le \; \tfrac{1}{(n+1)!}M(x-a)^{n+1}.$$

Had we started instead with the interval $[d,a]$, that is, with $x<a$ so that $x-a<0$, the n-fold integration would have given us

$$\big|f(x)-P_n(x)\big| \; \le \; \tfrac{1}{(n+1)!}M|x-a|^{n+1}. \qquad (3)$$

(Actually, when $x<a$ you'll get powers of $(a-x)$, and on the left-hand side you'll sometimes get $P(x)-f(x)$ instead of $f(x)-P(x)$. By writing the final result using absolute values, you

get a formula that covers both cases, $x > a$ and $x < a$. Consequently, form (3) is called the *standard form* of the Error Theorem.)

The key observation in all of this is that if you start with M on the right and do indefinite integration $n+1$ times, you get on the right (using x each time for simplicity instead of a new variable $u, v, \ldots$ each time)

$$M,\ Mx,\ \tfrac{1}{2}Mx^2,\ \tfrac{1}{3!}Mx^3, \ldots,\ \tfrac{1}{n!}Mx^n,\ \tfrac{1}{(n+1)!}Mx^{n+1}.$$

Exercises

1. Do the calculation from (1) to (2) again, this time using the interval $[d, 0]$ (that is, $x < a = 0$), and check that your final line, though not the same as (2), is still consistent with (3).

2. The argument on this handout actually proves more than the standard form of the Error Theorem. There is no reason why the lower bound has to be $-M$, where M is the upper bound.

a) Suppose that $U \le f'''(x) \le M$ for all $x \in [0, d]$. Do the argument from (1) to (2) over again to show that

$$\tfrac{1}{3!}Ux^3 \ \le\ f(x) - P_2(x) \ \le\ \tfrac{1}{3!}Mx^3. \tag{4}$$

b) Use (4) to show that, for $0 \le x \le 1$,

$$1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \ \le\ e^x \ \le\ 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}ex^3.$$