

## The Ratio Comparison Test

Recall that  $\sum \frac{1}{n^2-1}$  converges but that we can't prove it quickly by the Comparison Test because we have  $-1$  not  $+1$  in the denominator; that is,  $\frac{1}{n^2-1} > \frac{1}{n^2}$ . Andrew Quinton gave an intuitive argument why  $\sum \frac{1}{n^2-1}$  converges (the  $1$ , whether plus or minus, becomes negligible compared to the  $n^2$ ), and it would be nice if we could make this into a theorem so we could use it. There is such a theorem, but not in our book. It is the Ratio Comparison Test (not to be confused with the Ratio Test). Here is *one form* of it.

**Theorem.** (Ratio Comparison Test). Consider two series  $\sum a_n$  and  $\sum b_n$ . If

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right| = c > 0, \quad (1)$$

then either both  $\sum |a_n|$  and  $\sum |b_n|$  converge or both diverge. That is, if the absolute value of the ratio of corresponding terms in the two series has a finite limit, and it is greater than  $0$ , then  $\sum a_n$  converges absolutely if and only if  $\sum b_n$  converges absolutely.

*Example.* Let us show that the series  $\sum 1/(n^2-1)$  converges by doing a ratio comparison with  $\sum 1/n^2$ . We've noted that the Comparison Test fails here, so watch how negligible terms are no problem for the Ratio Comparison Test.

Since every term in both series is positive, we can ignore the absolute values in the ratio, obtaining

$$\frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \frac{n^2}{n^2-1} = \frac{1}{1 - (\frac{1}{n^2})} \rightarrow \frac{1}{1} = 1 > 0.$$

Therefore, since  $\sum 1/n^2$  converges absolutely ( $p$  test), so does  $\sum 1/(n^2-1)$ . Since absolute convergence implies convergence, we get that  $\sum 1/(n^2-1)$  converges. (In fact, in this case there is no difference between convergence and absolute convergence, since all terms are positive.)

*Idea of the proof of the Theorem.* Suppose  $c = .5$  and that  $\sum |b_n|$  converges. By (1), eventually  $|a_n|/|b_n| \leq .6$ , that is,  $|a_n| \leq .6|b_n|$ . Well, the series  $\sum .6|b_n|$  converges, so by the Comparison Test, so does  $\sum |a_n|$ . On the other hand, if  $\sum |b_n|$  diverges, we note that, by (1), eventually  $|a_n| \geq .4|b_n|$ . Well, the series  $\sum .4|b_n|$  diverges (because  $.4$  is not  $0$ ), so by the Comparison Test, so does  $\sum |a_n|$ .

### Problems

1. Determine whether or not the following converge.

a)  $\sum_{n=3}^{\infty} \frac{1}{\frac{1}{2}n^3 - n^2}$       b)  $\sum_{k=1}^{\infty} \frac{1}{k + \sqrt{k}}$

2. Consider  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k + \sqrt{k}}$ . Explain why its convergence or divergence *cannot* be determined by the Ratio Comparison Test. Then figure out whether it converges or not by some other test you know.
3. The Ratio Comparison Test isn't really necessary (though it sure is convenient). Every problem that is solvable using it can be solved by the ordinary Comparison Test — you just have to figure out a more subtle comparison.

Determine the convergence of the series in Problem 1b by the ordinary Comparison Test. *Hint:* Don't compare to  $1/k$ ; you need something slightly more complicated (not much).