C. (Obtaining the exponential generating function for derangements from the original recurrence) Let d_n be the number of derangements and let $D(x) = \sum_{n\geq 0} d_n x^n/n!$. We proved in class

$$d_{n+1} = n\,d_n + n\,d_{n-1}. \tag{1}$$

Roberts first gets another recurrence from (1) and then finds the exponential generating function. Here we get the exponential generating function directly from (1).

a) Using the argument in class, (1) is true for $n \geq 2$. Show how to make it true for $n \geq 1$. (This is good to do, because then there are fewer special terms when we turn (1) into a generating function equation.)

b) Multiplying (1) by $x^n/n!$ and summing, we get

$$\sum_{n\geq 1} d_{n+1}\frac{x^n}{n!} = \sum_{n\geq 1} n\,d_n\frac{x^n}{n!} + \sum_{n\geq 1} n\,d_{n-1}\frac{x^n}{n!}. \tag{2}$$

Show why (2) is the same thing as

$$D'(x) = xD'(x) + xD(x). \tag{3}$$

That is, (3) is a term by term translation of (2); I have not moved anything around. Showing this takes some care; you have to show that I haven't left out any initial terms, or if I have, they are 0.

c) Solve (3) for $D(x)$. This is a differential equation you can solve by methods of first-year calculus. You should get exactly the formula 1/3 down p 225 in Roberts (called $H(x)$ there). From there, Roberts shows how to get an explicit formula for d_n.