Let $G(X,Y)$ be a bipartite graph with parts X, Y. By a 2-to-1 complete matching of X to Y we mean a subgraph in which every $x \in X$ has degree 2 and every $y \in Y$ has degress ≤ 1. For any vertex v in any graph, $N(v) = \{u \mid uv \in E\}$ and $N(S) = \bigcup_{v \in S} N(v)$.

Theorem. A bipartite graph $G(X,Y)$ has a 2-to-1 complete matching from X to Y iff

$$\forall S \subset X, \quad |N(S)| \geq 2|S|. \tag{1}$$

Proof: We create another bipartite graph $G(\widehat{X}, Y)$ such that G has a complete 2-to-1 matching from X to Y iff $\widehat{G}$ has a complete matching from $\widehat{X}$ to Y. Then we show that Hall's condition on $\widehat{G}$ is equivalent to (1) on G.

First we define $\widehat{G}$: If $X = \{1, 2, \ldots, m\}$, then let

$$X^* = \{1^*, 2^*, \ldots, m^*\}$$
$$\widehat{X} = X \cup X^*.$$

$1^*, 2^*, \ldots, m^*$ are the *ideal elements*.

Next create edges from X^* to Y to duplicate the edges from X. That is, for all $i \in X$, arrange that $N(i^*) = N(i)$. It is clear that every complete 2-to-1 matching in G from X to Y now corresponds to a complete matching from $\widehat{X}$ in $\widehat{G}$: for each $i \in X$ just take either edge and move its X end to i^*. Similarly, given a complete matching from $\widehat{X}$ in $\widehat{G}$, for each edge ending at an i^*, move that end to the corresponding i.

To prove Condition (1), henceforth let S always denote a subset of X and T denote a subset of $\widehat{X}$. Then for all T define

$$\widehat{T} = \bigcup_{i \text{ or } i^* \in T} \{i, i^*\},$$
$$S_T = \bigcup_{i \text{ or } i^* \in T} \{i\}.$$

Note that $S_T \subset X$, and for all T there exists S such that $\widehat{S} = \widehat{T}$ (namely, $S = S_T$). Also note that

$$\forall T \subset \widehat{X}, \quad N(T) = N(\widehat{T}) \quad \text{and} \quad |\widehat{T}| \geq |T|. \tag{2}$$

Thus we claim

$$\forall T \subset \widehat{X}, \quad |N(T)| \geq |T| \tag{3}$$
$$\Longleftrightarrow$$
$$\forall \widehat{T} \subset \widehat{X}, \quad |N(\widehat{T})| \geq |\widehat{T}|. \tag{4}$$

Certainly (3)$\Longrightarrow$(4) since the $\widehat{T}$ sets are special cases of T sets; and (4)$\Longrightarrow$(3) by (2): for any T,

$$|N(T)| = |N(\widehat{T})| \overset{(4)}{\geq} |\widehat{T}| \geq |T|.$$

Finally, to prove (1), we show that it is equivalent to Hall's condition:

$$
\begin{aligned}
& X \text{ has a complete 2-to-1 matching in } G \\
& \iff \widehat{X} \text{ has a complete matching in } \widehat{G} \\
& \iff \forall T \subset \widehat{X}, \quad |N(T)| \geq |T| && \text{[Hall's Thm]} \\
& \iff \forall \widehat{T} \subset \widehat{X}, \quad |N(\widehat{T})| \geq |\widehat{T}| \\
& \iff \forall S \subset X, \quad |N(\widehat{S})| \geq |\widehat{S}| && \text{[every } \widehat{T} \text{ is an } \widehat{S}\text{]} \\
& \iff \forall S \subset X, \quad |N(S)| \geq 2|S|. \ \blacksquare && [|\widehat{S}| = 2|S|]
\end{aligned}
$$

Query to check your understanding:

Suppose $U = \widehat{T}$ for some T. Then for how many T is $U = \widehat{T}$?