

F. You toss a die until you get a 6. Let p_k be the probability that you get your first 6 on the k th toss. Use the method of convolving first occurrence with occurrence to find the generating function for $\{p_k\}$. Actually, do this using both variants mentioned in class, i.e., either setting $q_0 = 0$ or $q_0 = 1$, where q_k is the probability that you get *some* 6 on the k th toss. Either way, you should get $P(x) = \frac{x}{6 - 5x}$.

G. Prove the identity

$$\sum_{i=0}^t \binom{2i}{i} \binom{2t-2i}{t-i} = 4^t. \quad (1)$$

Hint: The generating function of $\{4^t\}$ is $\frac{1}{1-4x}$. *Note:* I would be very grateful if you can find a proof of (1) that does not involve generating functions. I have tried and tried to find a combinatorial argument.

H1. In the next several problems, let a_n be the number of H/T sequences of length n that end in the first HH. Long ago I announced that $A(x) = x^2/(1-x-x^2)$. You may use this fact without proof until further notice.

Find the generating function for the number of n -long sequences of H and T for which the second occurrence of HH finishes on the n th toss,

- a) if HHH is not considered to contain two occurrences of HH
- b) if HHH is considered to contain two occurrences of HH.

H2. Same as the previous problem, but now let the coefficients of the generating functions be probabilities instead of counts. I.e., under both assumptions use the probability that the second HH occurs on the n th toss. From the generating functions, also find the expected number of tosses to reach the second HH.

H3. Now prove that $A(x) = x^2/(1-x-x^2)$. Use the method that relates first occurrence to occurrence. (Be careful; this is trickier than the example in class of the first occurrence of a H because an event spans two tosses)

H4. Consider an infinite sequence of trials. Let f_n be the probability that some specific event happens (or finishes happening) for the first time on the n th trial. Let s_n be the probability that that event happens or finishes happening for the second time on the n th trial. Let k_n be the probability that that event happens or finishes happening for the k th time on the n th trial. Assume that occurrences of this special event cannot overlap. Let $F(x), S(x), K(x)$ be the associated ordinary generating functions.

- a) Relate $F(x)$ to $S(x)$.
- b) Relate $F(x)$ to $K(x)$.

Finally, let m_n be the probability that that event has not yet happened by the n th trial.

- c) Find the associated generating function $M(x)$.

- I.**
- a)** Consider a 4×4 chessboard on which you are allowed to put rooks in the top left 2×2 corner and the bottom right 2×2 corner. Compute the rook polynomial. (Get numbers for each coefficient.)
 - b)** Generalize to the $2n \times 2n$ board. (Your coefficients will be in terms of n .)