1. Find the maximum rate of change of \( f(x, y) = x^2 y + 2\sqrt{y} \) at the point \((2, 1)\). In which direction does it occur?

Ans 1. The max rate of change is always \(|\nabla f|\) in the direction \(\nabla f\), or \(\nabla f / |\nabla f|\) if you unitize. In this case \(\nabla f = (2xy, x^2 + 1/\sqrt{y})\), so \(\nabla f(2, 1) = (4, 5)\). Thus \(\sqrt{41}\) is the max rate and the direction is \((4, 5) / \sqrt{41}\).

2. Find the rate of growth of \( h(u, v) = u/v \) at the point \(p = (2, 1)\) in the direction from \(p\) to \((6, 4)\).

Ans 2. First, \(v = (6, 4) - (2, 1) = (4, 3)\). Unitizing, \(u = (4, 3) / \sqrt{25}\). Next \(\nabla h = (1/v, -uv^{-2}) = (1, -2)\) at \((u, v) = (2, 1)\). Thus \(\frac{\partial h}{\partial u} = \nabla h \cdot u = (1, -2) \cdot \frac{(4, 3)}{5} = -2/5\).

3. Suppose \(f(x, y, z) = xe^y + xz\), but in addition, \(y = \ln x\) and \(z = x^2\). If we define \(g(x) = f(x, y(x), z(x))\), find \(g'(e)\) two ways:

   a) By finding a formula for \(g\) and differentiating directly.

   b) By the chain rule.

Ans 3. a) \(g(x) = xe^{\ln x} + x(x^2) = x^2 + x^3\). Thus \(g'(x) = 2x + 3x^2\) and \(g'(e) = 2e + 3e^2\).

b) \[
\frac{dg}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx} = (e^y + z, xe^y, x) \cdot (1, 1/x, 2x) = (e^y + z) + e^y + 2x^2.
\]

When \(x = e\), we have \(y = \ln e = 1\) and \(z = e^2\). Thus \(g'(3) = (e + e^2) + e + 2e^2 = 2e + 3e^2\).

4. Find all critical points for \(f(x, y) = x^2 + ye^y\) and for each one, determine if it is a local max, local min, or a saddle point.
Ans 4. The critical points are where $\nabla f = (0, 0)$, so

$$2x = 0 \implies x = 0,$$

and

$$1 e^y + ye^y = (y + 1)e^y = 0 \implies y + 1 = 0 \implies y = -1.$$  

So the only critical point is $\mathbf{x} = (0, -1)$. Now do the second order test:

$$f_{xx} = 2 > 0,$$

$$f_{xy} = 0,$$

$$f_{yy} = 1 e^y + (y + 1)e^y = (y + 2)e^y,$$

so when $y = -1$, then $f_{yy} = 1/e$.

Thus at the critical point, $D = 2 \cdot (1/e) - 0^2 > 0$. Since both $f_{xx}$ and $D$ are positive, there is a minimum at $(0, -1)$.