1. Using the formula is S13.4, compute
   a) $\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}$
   b) $\begin{vmatrix} 1 & 1 \\ 6 & 2 \end{vmatrix}$
   c) $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$

2. By brute force evaluation, show that
   $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   and
   $\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

   The result is true more generally: for any size determinant, if you multiply a single row by $k$, you multiply the determinant by $k$.

3. Show that $\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. What do you suppose is the general principle here?

4. Write an identity relating $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix}$. In light of previous problems, you should be able to figure out the answer without any computations, but computations are ok.

5. Show that $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. The result is true more generally: for any size determinant, if you switch two rows, you change the sign of the determinant.

6. Guess the relationship between $\begin{vmatrix} b & a \\ d & c \end{vmatrix}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Show that you are right.

7. Compute $\begin{vmatrix} a & b \\ a & b \end{vmatrix}$ and $\begin{vmatrix} a & a \\ c & c \end{vmatrix}$. Could you have guessed the answers based on previous problems?

8. The matrix $M = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is said to be the transpose of $N = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$; note that each is obtained from the other by flipping entries around the main diagonal $a$-$d$. Show that $|M| = |N|$. Again, this equality holds for any size determinants.

9. Verify that $\begin{vmatrix} 1 & x & y \\ 1 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 0$ is the equation of the straight line through the points $(1,2)$ and $(4,-1)$. (This is not an accident.)

10. Your book defines $3 \times 3$ determinants by a first row expansion formula:

    $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

    Use this formula (and previous problems) to prove that if any two rows of a $3 \times 3$ determinant are the same, then det $= 0$. You’ll need to consider two cases: either the first row is not one of the equal rows, or it is.
11. Evaluate
\[
\begin{vmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{vmatrix}
\quad \begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
5 & 4 & 3
\end{vmatrix}
\]

12. Show that the following first column expansion formula gives the same value to $3 \times 3$ determinants as the first row expansion formula. (There are in fact expansion formulas for each row and each column, but the signs of the terms do not always follow the same pattern.)
\[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} = a_1 \begin{vmatrix}
b_2 & b_3 \\
c_2 & c_3
\end{vmatrix} - b_1 \begin{vmatrix}
a_2 & a_3 \\
c_2 & c_3
\end{vmatrix} + c_1 \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3
\end{vmatrix}.
\]

13. Compute
\[
\begin{vmatrix}
a_1 & a_2 \\
0 & b_2
\end{vmatrix}
\quad \begin{vmatrix}
a_1 & a_2 & a_3 \\
0 & b_2 & b_3 \\
0 & 0 & c_3
\end{vmatrix}
\]

What’s the pattern?

14. Consider the vectors $(3,1)$ and $(1,2)$, drawn as arrows with both tails at the origin. Let $P$ be the parallelogram with these vectors as two of the sides.

a) Compute the area of $P$. (This is a bit tricky, but there are several ways to do it.)

b) Compute $\begin{vmatrix}
3 & 1 \\
1 & 2
\end{vmatrix}$.

c) Guess a general conclusion.

15. The text proves that if $\mathbf{u}, \mathbf{v}$ are the sides of a parallelogram in $\mathbb{R}^3$, then $|\mathbf{u} \times \mathbf{v}|$ is the area of that parallelogram. Show that the result in Problem 14c follows from this theorem. Hint: Vectors in $\mathbb{R}^2$ can be viewed as vectors in $\mathbb{R}^3$ by adding a third coordinate. Specifically, view $(3,1)$ and $(1,2)$ as $\mathbf{u} = (3,1,0)$ and $\mathbf{v} = (1,2,0)$.

16. Consider the determinant
\[
\begin{vmatrix}
2 & 2 & 1 \\
2 & -1 & -2 \\
-1 & 2 & -2
\end{vmatrix}
\]

a) Verify that the rows of this determinant are mutually perpendicular vectors. Thus, if we were to draw all three vectors as arrows with their tails at the origin, we would get edges of a rectangular box.

b) Figure out the lengths of the sides of this box and thus its volume.

c) Compute the determinant.