Below, every vector quantity is a function of $t$ (usually time), but we “suppress” the $t$. That is, we write $r$ instead of $r(t)$, and so forth.

$$r' = \mathbf{v}$$  \hspace{1cm} (The derivative of position is velocity)

$$|\mathbf{v}| = v$$  \hspace{1cm} (The magnitude of velocity is speed)

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{v}$$  \hspace{1cm} (The unit tangent vector is the unitized velocity vector)

$$s = \int v \, dt$$  \hspace{1cm} (arc length is the integral of speed along the curve)

so $s' = v$ (the derivative of arc length is speed)

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$  \hspace{1cm} (The normal vector to the curve is the unitized derivative of $\mathbf{T}$)

so $\mathbf{T}' = |\mathbf{T}'| \mathbf{N}$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}}{ds/dt} \right| = \frac{|\mathbf{T}'|}{v}$$  \hspace{1cm} (Curvature is the magnitude of how fast the tangent changes direction as you move along the curve at unit speed.)

Thus $|\mathbf{T}'| = \kappa v$

and $\mathbf{T}' = |\mathbf{T}'| \mathbf{N} = \kappa v \mathbf{N}$.

$$\mathbf{v} = v \mathbf{T}$$

so differentiating, $\mathbf{a} = v' \mathbf{T} + v \mathbf{T}' = v' \mathbf{T} + \kappa v^2 \mathbf{N}$

(Acceleration, as a vector quantity, has both a tangential and a normal component.)

Note: Except for the indented equations that begin with “so” or “thus”, these equations are essentially definitions. For instance, in the first equation, we declare $r'$ to be called velocity, because it has properties that we would expect a vector concept of velocity to have. Definitions don’t have proofs. They just have motivations.

The last equation contains both $\mathbf{a}$ and $v'$. The latter is the rate of change of speed. In first-year calculus, you might think of this as acceleration, but what this equation says is that in several variables this “scalar acceleration” is very different from the vector quantity acceleration.