Algorithms and Complexity

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procedure \( mst(x:\text{array of } n \text{ points in the plane}) \);
{constructs a spanning tree \( T \) of minimum length, on the}
vertices \( \{x_1, x_2, \ldots, x_n\} \) in the plane\}
let \( T \) consist of a single vertex, \( x_1 \);
while \( T \) has fewer than \( n \) vertices do
for each vertex \( v \) that is not yet in \( T \), find the
distance \( d(v) \) from \( v \) to the nearest vertex of \( T \);
let \( v^* \) be the vertex of smallest \( d(v) \);
adjoint \( v^* \) to the vertex set of \( T \);
adjoint to \( T \) the edge from \( v^* \) to the nearest
vertex \( w \neq v^* \) of \( T \);
end{while}
end. \( \{mst\} \)

**Proof of correctness of \( mst \):** Let \( T \) be the tree that is produced by
running \( mst \), and let \( e_1, \ldots, e_{n-1} \) be its edges, listed in the same order in
which algorithm \( mst \) produced them.

Let \( T' \) be a minimum spanning tree for \( x \). Let \( e_r \) be the first edge of \( T \)
that does not appear in \( T' \). In the minimum tree \( T' \), edges \( e_1, \ldots, e_{r-1} \) all
appear, and we let \( S \) be the union of their vertex sets. In \( T' \) let \( f \) be the
edge that joins the subtree on \( S \) to the subtree on the remaining vertices
of \( x \).

Suppose \( f \) is shorter than \( e_r \). Then \( f \) was one of the edges that was
available to algorithm \( mst \) at the instant that it chose \( e_r \), and since \( e_r \) was
the shortest available edge at that moment, we have a contradiction.

Suppose \( f \) is longer than \( e_r \). Then \( T' \) would not be minimal because
the tree that we would obtain by exchanging \( f \) for \( e_r \) in \( T' \) (why is it
still a tree if we do that exchange?) would be shorter, contradicting the
minimality of \( T' \).

Hence \( f \) and \( e_r \) have the same length. In \( T' \) exchange \( f \) for \( e_r \). Then
\( T' \) is still a tree, and is still a minimum spanning tree.

The index of the first edge of \( T \) that does not appear in \( T' \) is now at
least \( r + 1 \), one unit larger than it was before. The process of replacing
dges of \( T \) that do not appear in \( T' \) without affecting the minimality of
\( T \) can be repeated until every edge of \( T \) appears in \( T' \), i.e., until \( T = T' \).
Hence \( T \) was a minimum spanning tree. \( \blacksquare \)