In the solutions, if I write something like $x = .563$ what I mean is that .563 is the value of $x$ to 3-decimal accuracy, which I computed with a calculator.

1. Find the angle between $(1,2)$ and $(3,-1)$.

   **Ans 1.** $|{(1,2)}||{(3,-1)}| \cos \theta = (1, 2) \cdot (3, -1) = 1$. Thus
   
   \[ \cos \theta = \frac{1}{\sqrt{5} \sqrt{10}} = \frac{1}{\sqrt{50}} = .141 \]
   
   \[ \theta = \cos^{-1} \left( \frac{1}{\sqrt{50}} \right) = 81.87^\circ. \]

2. Suppose the half life of radioactive iodine in a tracer medical injection is 2 hours. Suppose the amount of this iodine left in you 3 hours after an injection is 1 mg. Using only mental computations, determine how much is left in you 7 hours after the injection. Explain your method briefly.

   **Ans 2.** Because 2 hours is the halflife, after 2 more hours (5 hours total) .5mg is left, and after another 2 hours (7 total), .25mg is left.

3. What’s wrong with the next sentence, no matter what function $f$ is?

   The rate of growth of $f(x, y)$ at $(1,2)$ is 4.

   **Ans 3.** Functions of 2 variables don’t have one rate of growth, since there are infinitely many directions you could move from the base point. It would be correct to talk about the max rate of growth, or the growth rate in a particular direction.

4. Amtrak and the airlines compete for passengers on the New York to Washington route. The number of people who take the train is a function of both fares – train and plane. Let $N(x, y)$ be the number of people who take the train when the train fare is $\$x$ and the airfare is $\$y$.

   a) Interpret carefully the partials $N_x$ and $N_y$.

   b) Use common sense to determine the sign of $N_x$; of $N_y$. Explain briefly.

   **Ans 4.** a) To be careful, it will help to refer to the point at which the partials are being evaluated, so I do that at first.

   $N_x(t,p)$ is the number of extra NYC–DC train passengers per dollar as the train fare goes up from $\$t$ while the plane fare remains fixed at $\$p$. One typically says that $N_x$ is the number of extra train passengers (or the marginal number of train passengers) if the train fare goes up by $\$1$ from $\$t$ while the airfare is held constant at $\$p$. 

   b) Use common sense to determine the sign of $N_x$; of $N_y$. Explain briefly.
Similarly, $N_y$ is the number of additional NYC–DC train passengers if the plane fare goes up by $\$1$ while the train fare remains the same.

b) $N_x < 0$ and $N_y > 0$. If the train fare goes up while the plane fare remains the same, the train becomes less attractive — some people switch to the plane. But if the plane fare goes up while the train fare remains the same, people switch to the train, and $N(x, y)$ counts the number of train riders.

5. Sketch a possible contour diagram for a rectangular plot of land on which there is one hill in the middle, which is very steep on the west side and has a gentle rise on the other. Otherwise the land is pretty flat, except that it falls off steeply in the northeast corner.

Ans 5.

![Contour Diagram]

6. Consider the function $g(x, y) = x^2 + 2y^2$.

a) Find the critical point (there is only one).

b) Use the 2nd-order test to determine if that critical point is a max point, min point, or neither.

c) Explain briefly how for this function you could obtain the answers to a) and b) in your head without any calculus.

Ans 6.

a) Set

\[
g_x = 2x = 0
\]

\[
g_y = 4y = 0.
\]

Clearly the only solution is $x = y = 0$.

b) The 2nd-order test involves the signs of $g_{xx}$ and $D = g_{xx} g_{yy} - (g_{xy})^2$ at the critical point. Also,

\[
g_{xx} = 2 \text{ everywhere}
\]

\[
g_{yy} = 4 \text{ everywhere}
\]

\[
g_{xy} = 0 \text{ everywhere}.
\]

Thus at the critical point, $g_{xx} > 0$ and $D = 8 > 0$. Thus $(0, 0)$ satisfies the test to be a minimum point.

c) A sum of squares is always positive, except if all the squares are 0, which happens here only when $x = y = 0$. Thus $(0, 0)$ is not just a minimum point, but the only minimum point.
7. Estimate \( f_y(1, 2) \) and \( f_x(2, 1) \) from the contour map below. Explain your computation briefly.

\[
\begin{align*}
\text{Ans 7. } f(1, 2) & \approx 26.5 \text{ and moving half a unit up (down) } y \text{ direction gets you to the next lower (higher) integer output, so } f_y(1, 2) = -1. \text{ Symbolically, } \\
\frac{f(1, 2.5) - f(1, 2)}{2.5 - 2} & \approx \frac{26 - 26.5}{.5} = -1 \quad \text{or} \quad \frac{f(1, 1.5) - f(1, 2)}{1.5 - 2} \approx \frac{27 - 26.5}{- .5} = -1.
\end{align*}
\]

As usual, a 2-sided approximation is fine too.

As for \( f_x(2, 1) \), eyeballing suggests that the right and left approximations differ, so a 2-sided approximation is probably better, though I will accept any of them. So I compute

\[
\frac{f(3.7, 1) - f(.7, 1)}{3.7 - .7} = \frac{26 - 28}{3} = - .667
\]

8. Let \( f(x, y) = x^2 e^{-y} \). Let \( a = (1, 0) \).

\[(2)\] Evaluate \( f(a) \).
\[(6)\] Compute \( \nabla f(a) \).
\[(5)\] Use your knowledge of \( f(a) \) and a standard approximation technique to approximate \( f(1.1, -.2) \). Compare with the exact value as computed on your calculator.
\[(4)\] What is the rate of growth of \( f \) from \( a \) in the normalized direction of \((1, 1)\)?
\[(4)\] What is the direction of most negative rate of growth of \( f \) from \( a \), and what is that rate of growth?

\[(8)\] a) \( 1^2 e^0 = 1. \)

b) \( f_x(x, y) = 2xe^{-y} \) and \( f_y(x, y) = -x^2 e^{-y} \), so \( f_x(1, 0) = 2 \) and \( f_y(1, 0) = -1 \). Thus \( \nabla f(a) = (2, -1) \).

c) Using the tangent plane approximation,

\[
f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y) \Delta x + f_y(x, y) \Delta y.
\]

Set \( (x, y) = a \) and \( (\Delta x, \Delta y) = (.1, -.2) \). Then (1) becomes

\[
f(1.1, -.2) \approx f(a) + (2, -1) \cdot (.1, -.2) = 1 + .2 + .2 = 1.4.
\]
Computing $f(1.1, -2)$ directly with a calculator yields 1.478.

d) The normalized vector is $(1, 1)/\sqrt{2}$, so the growth rate is
\[ \nabla f(a) \cdot (1, 1)/\sqrt{2} = 1/\sqrt{2} = .707 \]

e) It’s a theorem that this direction is $-\nabla f$ (or $-\nabla f/|\nabla f|$ if you want to normalize before computing the rate) and the growth rate is $-|\nabla f|$. Thus at $a$, the direction is $(-2, 1)$ (or $(-2, 1)/\sqrt{5}$) and the rate is $-\sqrt{5}$.

9. a) State the general solution $y(t)$ to
\[ \frac{dy}{dt} = -2(y - 3). \]  

(2)

b) Verify that your answer to a) is a solution to (2).

c) Find the particular solution if $y = 25$ when $t = 0$.

d) For this particular solution, what is $y(2)$?

e) Make up a plausible word problem for which (2) is the right model.

Ans 9. a) $y(t) = 3 + Ce^{-2t}$.

b) Plug into both sides of (2):
\[ \frac{dy}{dt} = -2Ce^{-2t}, \]

and
\[ -2(y - 3) = -2((3+Ce^{-2t}) - 3) = -2(Ce^{-2t}) = -2Ce^{-2t}. \]
The sides agree.

c) $25 = y(0) = 3 + Ce^0 = 3 + C$, so $C = 22$. Thus the particular solution is
\[ y(t) = 3 + 22e^{-2t}. \]

d) 
\[ y(2) = 22e^{-4} + 3 \]
\[ = .403 + 3 \]
\[ = 3.403. \] [calculator]

e) There are lots of different plausible stories, but good stories must all correctly make use of the specific coefficients in the differential equation, namely $-2$ and $3$.

In the form given, $y' = -2(y - 3)$, the equation most easily models a heating/cooling problem, say: An item is left out on a cold winter night with ambient temperature $3^\circ$ (say Celsius). What is the general formula for the temperature of the object at time $t$ (say in hours) if it cools off very fast, at an instantaneous rate of twice the difference from the ambient temperature per time period (hours).
The differential equation can be restated as \( \frac{dy}{dt} = 6 - 2y \). In this form it is the model for a patient taking a drug intravenously (continuously) at the rate of 6 drug units per time unit (say 6mg/day) when the body purges the drug instantaneously at a rate of twice the full amount per time period.

Instead of a drug in a person, \( y(t) \) could be pollution in a lake, with a factory putting it in and natural flushing taking it out.

10. Evaluate \( \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \).

Ans 10. Let \( u = \sqrt{x} \), then \( du = dx/(2\sqrt{x}) \). Thus
\[
\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \cos \sqrt{x} \frac{dx}{2\sqrt{x}} = 2 \int \cos u \, du = 2 \sin u + C = 2 \sin \sqrt{x} + C.
\]

11. The function \( p(x) = 3x^2 \) is a probability density function if \( x \) is restricted to the interval \([0, 1]\).

(a) Verify this. (Begin by stating what has to be verified.)

(b) Sketch in your blue book a graph of \( p(x) \) over the interval \([0, 1]\) and use this to visually estimate the median of the distribution. Draw enough on your graph to explain how you did your estimate.

(c) Compute the median exactly. Hopefully your estimate was close, but don’t change it in any event.

(d) Compute the mean exactly.

Ans 11. (a) Since \( p(x) \) is clearly never negative, it only needs to be verified that \( \int_0^1 p(x) \, dx = 1 \).

In fact
\[
\int_0^1 3x^2 \, dx = x^3 \bigg|_0^1 = 1 - 0 = 1.
\]

(b) I estimate \( 3/4 \) for the median. The area seems to be equal on both sides of the vertical line I have drawn.

\[
.5 = \int_0^m 3x^2 \, dx = m^3 - 0^3
\]
\[
m \approx \sqrt[3]{.5} \approx .794
\]
Solution to Final Exam

\[ \int_0^1 x p(x) \, dx = \int_0^1 3x^3 \, dx = .75x^4 \bigg|_0^1 = .75. \]

Advanced Problems.

12. In class we learned the following:

Suppose a drug is given in regular discrete doses of amount \( a \). Suppose \( r \) is the fraction left by the time of the next dose. Then

\[ \frac{a}{1 - r} \]

is the long-run maximum amount in the body.

But this formula came at the end of a long development. First we had to look at the amount in the body after 1, 2, \ldots doses. We had to distinguish between before and after doses, and inbetween. Then we had to figure out if there was a limit and how to express it. Finally, we had to figure out if this limit represented a steady state or part of a steady cycle, or what.

In your own words (and symbols!) carry out this whole-development in a 30-minute essay, starting with the real-world situation and ending with \( \frac{a}{1 - r} \). Of course, it's all there in the book, but you have distill it and show that you understand it.

Ans 12. You need to work through all of the following, except perhaps the last, in your own way:

- What the amount in the body is just before and after the 1st dose, the 2nd dose, etc., obtaining after algebraic simplification a finite geometric series \( a + ar + \cdots + ar^n \).

- A proof that \( a + ar + \cdots + ar^n = a(1 - r^{n+1})/(1 - r) \).

- An explanation why \( a(1 - r^{n+1})/(1 - r) \rightarrow a/(1 - r) \) for \( r \) values that occur in medical situations, so that it makes sense to talk about a "long-run" amount, and that \( a/(1 - r) \) is it. (If you introduce the infinite series \( a + ar + \cdots \), you will have to explain how an infinite sum can make sense.)

- Why \( a/(1 - r) \) is a long-run maximum that the actual amount in the body rises to, and to what extent it makes sense in this discrete problem to talk about a steady state. In this context it will help to talk about long-run minimum as well.

- Perhaps the quick proof (without geometric series) that if there is a long run maximum, it has to be \( a/(1 - r) \).

Both calculations and words will be necessary to treat several of these items, and graphs will be helpful for some.

13. The following problem comes from an engineering student at another school who had been asked to solve it one summer while working for an oil company.
When crude oil flows from a well, water is frequently mixed with it in an emulsion. To remove the water the crude is piped to a device called a heater-treater, which is simply a large tank in which the oil is warmed and the water is allowed to settle out. Operating experience in a particular oil field indicates that the concentration \( C \) of water in the treaters's output can be modeled by the following equation in a neighborhood of the usual operation point of 135°F Fahrenheit and a 2-hour holding time:

\[
C = 0.04 - 0.0032h^2 - 10^{-6}t^2 + 4.1 \times 10^{-5}ht,
\]

where \( h \) is the holding time in hours and \( t \) is the operating temperature in degrees Fahrenheit.

a) Because of random fluctuations in the well's flow rate, the holding time actually varies slightly around 2 hours. Suppose you are given a simple control device that can change the tank temperature proportionally to the measured change in holding time. What constant of proportionality would best compensate for small holding-time fluctuations and keep the water concentration as constant as possible?

b) Now as field equipment ages, its maximum operating settings are generally decreased. Find the equation of the line that best approximates the way in which the holding time would have to be increased as the maximum temperature rating falls slightly below the usual operating temperature.

This problem is an example of something I mentioned in class: applied problems often come with a lot of words, but when a mathematician (you) finally figures out what is being asked, it turns out to be something basic and the computation is straightforward, though perhaps tedious because of the ugly numbers.

Ans 13. For both parts of the problem, the mathematical situation is the following: We are given Concentration as a function of holding time in hours and temperature in Fahrenheit

\[
C(h, t) = 0.04 - 0.0032h^2 - 10^{-6}t^2 + 4.1 \times 10^{-5}ht,
\]

and we want to hold \( C \) constant at the standard value \( C(2, 135) \). However, in each part of the problem there is one variable we can't control, so we have to compensate with the other. Ideally, we would compensate so that we stayed exactly on the contour curve (constraint set) \( C = C(2, 135) \). But there is another restriction: Out in the field we only have simple control devices that can merely keep us on straight lines in \((h, t)\) space through the point \((2, 135)\). The problem asks us for the best straight lines, that is, closest to the contour curve near \((2, 135)\). Tangent lines! Derivatives and partial derivatives!

There are several ways to go from here. Below we take the tangent plane approach to \( C(h, t) \). Another approach for part a) is to regard \( t \) as an implicit function of \( h \) in the equation \( C(2, 135) = C(h, t(h)) \).

a) We want to keep \( C \) constant as \( h \) varies slightly from 2. Slight changes are best modeled by approximating the function by its tangent plane:

\[
C(h, t) \approx C(2, 135) + \frac{\partial C}{\partial h}(2, 135) \Delta h + \frac{\partial C}{\partial t}(2, 135) \Delta t.
\]
We want to move along this tangent plane only with values of $\Delta h$ and $\Delta t$ such that
\[
\frac{\partial C}{\partial h}(2, 135) \Delta h + \frac{\partial C}{\partial t}(2, 135) \Delta t = 0.
\]
Our only control for doing this is to set
\[
\Delta t = k \Delta h,
\]
where we get to set $k$, because the problem says we can “change the tank temperature proportionally to the measured change in holding time.” That is, $\Delta h$ is out of our control, but we can control $\Delta t$ by setting $\Delta t = k \Delta t_{\text{ah}}$, where we also get to set $k$, but once and for all. Thus we want to solve for $k$ in
\[
\frac{\partial C}{\partial h}(2, 135) \Delta h + \frac{\partial C}{\partial t}(2, 135) k \Delta h = 0
\]
and so
\[
k = -\frac{\frac{\partial C}{\partial t}(2, 135)}{\frac{\partial C}{\partial h}(2, 135)}
\]
Since
\[
\frac{\partial C}{\partial h} = -0.0064h + 4.1(10^{-5}t)
\]
and
\[
\frac{\partial C}{\partial t} = -2(10^{-6}t) + 4.1(10^{-5}h)
\]
we get from (4) that
\[
k = -\frac{-7.26510^{-3}}{-1.88010^{-4}} = -38.6
\]
In other words, for every extra hour the oil sits in the tank, we should adjust the temperature gauge down by 38.6 F°. (I get the impression that the holding time only varies by small fractions of an hour, so you wouldn’t be changing the temperature setting very much.)

b) The only difference mathematically between parts a) and b) is that now the temperature setting becomes the variable we can’t control, and the holding time becomes the variable we can control in response (rather than vice versa). But the goal is the same: to relate $\Delta t$ and $\Delta h$ proportionally so that $\Delta C = 0$. The slope $m$ will be the reciprocal of $k$ (more detail below). The only other change is that we are asked for the equation of the line, not just the value of $m$. Well, the known point on the tangent line is $(2, 135)$, so the line is
\[
h = 2 - 0.0259(t - 135) \quad \text{or} \quad h = 2 + 0.0259(135 - t) = 5.497 - 0.0259t.
\]

Why is $m = 1/k$? Well, we want the change in $h$ per change in $t$, rather than vice versa. In more detail, we set $\Delta h = m \Delta t$ and thus rewrite (3) as
\[
\frac{\partial C}{\partial h}(2, 135) m \Delta t + \frac{\partial C}{\partial t}(2, 135) \Delta t = 0
\]
so
\[
m = -\frac{\frac{\partial C}{\partial t}(2, 135)}{\frac{\partial C}{\partial h}(2, 135)} = -\frac{1}{k} = -\frac{1}{-38.6} = 0.0259
\]

— end —