Instructions. Put your work in your bluebook(s) or, when requested, on the sheet with figures. The point count of each problem part is to its left. There are 11 problems worth 150 points total. You have 3 hours. Do not spend a lot of time on parts worth only a few points. Calculators are welcome, but indicate when you have used one, by writing “calculator”.

1. Plants need varying amounts of water and sunlight to grow. Each contour diagram below shows the growth of a different plant as a function of the amount of water and sunlight. Match the following descriptions with the right diagram; no explanation necessary.

a) A plant that does best in the desert (little water and lots of sunlight).

b) A plant that does best in a tropical rain forest (lots of water, little sunlight).

c) A plant that does best in a temperate climate (moderate water, moderate sunshine).

2. Americans go out and eat a lot of burgers and fries (usually together) or pizza, and the sales of each is sensitive to the average price of all of them. Let $B(b, f, p)$ be the number of burgers Americans buy per day if the average price is $b$ dollars for a burger, $f$ dollars for a serving of fries, and $p$ dollars for a pizza. Similarly, $F(b, f, p)$ is the number of servings of fries per day, and $P(b, f, p)$ is the number of pizzas.

a) Interpret $B_b$, that is, $\frac{\partial B}{\partial b}(b, f, p)$.

b) For each of the following, decide if you think it is positive or negative, and explain why briefly.

$B_b, B_f, B_p, P_b$. 
3. Trout are introduced to a stream. Trout are predators that eat other fish. The figure below shows how the trout population and the other fish population vary over time. (All other fish are lumped together.) The arrow shows the direction of time.

(3) a) What happens in the long run?

(8) b) On the same axes (horizontal time, vertical population), sketch the plot of both fish populations, trout and other, for several cycles.

4. A table of values for a function \( f(x, y) \) is given below.

(4) a) It happens that \( f_y(x, y) \) has the same sign throughout. What is that sign, and how do you know it is the same throughout?

(6) b) Estimate \( f_x(6, 10) \). Show how you obtain your estimate; even if your answer is wrong, you can get partial credit if your method is ok and I can follow it.

\[
x
\begin{array}{cccccccc}
    0 & 500 & 510 & 525 & 560 & 590 & 640 \\
    5 & 440 & 450 & 470 & 500 & 540 & 610 \\
\end{array}
\]

\[
y
\begin{array}{cccccccc}
    10 & 410 & 420 & 445 & 480 & 520 & 575 \\
    15 & 390 & 405 & 430 & 460 & 490 & 525 \\
    20 & 375 & 385 & 410 & 435 & 475 & 500 \\
\end{array}
\]

5. On the last page of this test are two copies of a graph with slope marks. Put your name on this sheet, draw your answers to this question on it, and submit it with your bluebook.

(6) a) In the first copy, assume the graph is a gradient field (with the arrowheads missing) for some function \( z = f(x, y) \). Draw in 3 rather different contour lines
for the function.

b) In the 2nd copy, assume the graph is a slope field. Draw in 3 rather different solutions to the associated differential equation.

c) Continuing with the slope field interpretation, which of the following differential equations could this be the slope field of?

\[ \frac{dy}{dx} = y, \quad \frac{dy}{dx} = \frac{x}{y}, \quad \frac{dy}{dx} = x^2 + y^2, \quad \frac{dy}{dx} = x - y. \]

6. The distribution of the heights \( x \) in meters of a group of shrubs is represented by the density function \( p(x) \) below. No shrubs are higher that 1.5 meters. Calculate the percentage of shrubs which are:

a) Less than 0.5 meters high.

b) Between 0.5 and 1 meter high.

c) More than 1 meter high.

\[ \begin{array}{c}
\text{Density of Shrub heights} \\
0.8 \\
0.5 \quad 1 \quad 1.5 \\
p(x)
\end{array} \]

7. Consider the function \( f(x, y) = x^2 + 3xy - 12y \). Find the critical point or points. You need not determine if they are max, min, or saddle points.

8. Let \( h(x, y) = \ln(xy^2) \).

a) Evaluate \( \nabla h(1, 1) \).

b) Determine \( \frac{\partial h}{\partial u} (1, 1) \) in the normalized direction \( u = \frac{1}{\sqrt{5}}(2, 1) \).

c) Consider \( d \), the directional derivative of \( h \) at \( (1, 1) \) in the normalized gradient direction. Which should be larger, \( d \) or the answer to b)? Explain how you know without computing \( d \). Then compute \( d \) and compare.
9. Let $S$ be the unit square, $0 \leq x \leq 1, 0 \leq y \leq 1$. Let $f(x, y) = x + 2y$.

(a) Find the constant $k$ so that $\int_S k f(x, y) \, dy \, dx = 1$.

(b) Why would one want to find such a $k$? That is, what special interpretation would the function $k f(x, y)$ then have?

(c) Divide $S$ into triangles like this: $\square$. Call the triangles $U$ (upper) and $L$ (lower).

Without actually computing anything, decide which integral is greater, $\int_U f(x, y) \, dy \, dx$ or $\int_L f(x, y) \, dy \, dx$.

Explain your reasoning briefly. (No erasing if part d makes you change your mind.)

(d) Compute whichever integral you thought was larger.

10. In this problem we compare two methods of administering drugs, once a day versus continuously (intravenously). They require different mathematical models, as well as different medical procedures. Below, $t$ is always measured in days, starting at $t = 0$ at the moment the treatment starts. In both models, assume the patient has none of this drug in him when the treatment starts.

(a) Daily model. We give a patient 10 mg of a drug at the start of every day. During each day, 33% of the amount in his body is purged. Determine the long-term effect — whatever you can say about a pattern or steady state for the amount $D$ of drug in his body if the treatment is continued. "Determine" means find the answer, but also carry out the mathematical analysis that leads to the answer.

(b) Continuous model. We give a patient the same drug intravenously, at a constant rate so that the amount administered every 24 hours totals 10 mg. At all times, his body purges this drug at an instantaneous rate of 40%. Determine the long-term effect — whatever you can say about a pattern or steady state for the amount $D$ of drug in his body if the treatment is continued.
11. The production function for a company is \( P = 200x^{3/4}y^{1/4} \), where \( P \) is the amount of its product produced given \( x \) units of labor and \( y \) units of equipment. Each unit of labor costs $1000 and each unit of equipment costs $300.

a) Assume the goal of the company is to maximize production given a fixed budget of $50,000.

i) What is the function to be optimized?

ii) What is the constraint equation?

iii) What are the equations to solve to determine the optimum? Write down the equations, but don’t solve them.

iv) What is the meaning of the Lagrange multiplier \( \lambda \) in this situation? (Recall: \( \lambda \) always has an interpretation. Although you can’t know its value without solving the optimization equations, you can know its units and interpretation without knowing its value.)

b) Now assume instead that the goal of the company is to minimize cost given a fixed production requirement of 8000 units. (For instance, it may be under contract with the government to produce 8000 units.) Answer i)–iv) again with this assumption.

**Bonus.** For Problem 6 about shrubs,

a) Find the median shrub height,

b) Find the mean shrub height.
Your Name __________________________