Problems on Vectors and Directional Derivatives

1. Let \( \mathbf{a} = (3, 4), \mathbf{u} = (2, -1) \). Compute
   a) \( \mathbf{a} + \mathbf{u} \)
   b) \( 3\mathbf{a} \)
   c) \( \mathbf{a} \cdot \mathbf{u} \)
   d) \( |\mathbf{u}| \)
   e) \( \mathbf{a} - \mathbf{u} \) (We didn’t define vector subtraction in class, but you can guess how to do it.)
   f) Make a rough sketch of all the points \( \mathbf{a} + t\mathbf{u} \), where \( t \) varies over the real numbers.
   g) The unitized (also called normalized) version of \( \mathbf{u} \). This is the positive scalar multiple \( k\mathbf{u} \) of \( \mathbf{u} \) whose length is 1. So you have to figure out the right \( k \) to get \( |k\mathbf{u}| = 1 \).

2. Make two pictures for the addition
   \( (2, 1) + (1, -3) = (3, -2) \),
   a) Representing every vector as an arrow.
   b) Representing some vectors as points.

3. The set of points \( (0, 1) + t(2, 1) \) is a line on the plane.
   a) Name a point on this line.
   b) Name a direction vector for this line.
   c) What is the equation for this line in the traditional form \( y = mx + b \)?

4. Let \( f(x, y) = 3x^2y \). Compute
   a) \( \nabla f(1, 2) \) and \( \nabla f(x, y) \).
   b) The equation of the tangent plane to (the graph of) \( f \) at (1,2).
   c) The exact value of \( f(1.1, 1.9) \).
   d) The approximate value of \( f(1.1, 1.9) \) using the plane that is tangent to \( f \) at (1,2). That is, compute the \( z \) value on this plane at the nearby point \( (x, y) = (1.1, 1.9) \). This computation is the multivariate analog of using the tangent line to a first-year calculus function to approximate the value of the function at a nearby point.
   e) \( (\nabla f(1, 2)) \cdot (3, 4) \)
   f) The directional derivative of \( f \) at (1,2) in the direction \( \mathbf{u} = (2, -1) \).
   g) Let \( g(x) \) be the \( x \)-slice of \( f \) through \( (x, y) = (1, 2) \), that is, the function with \( y \) held fixed. Write down the formula for \( g(x) \) and then compute \( g'(1) \). What quantity computed earlier in this problem must \( g'(1) \) equal?

5. Continuing with \( f(x, y) = 3x^2y \), compute
   a) \( \frac{\partial f}{\partial \mathbf{u}}(1, 2) \) when \( \mathbf{u} = (1, -2) \),
b) $\frac{\partial f}{\partial u}(1, 2)$ when $u = (2, -4)$.

c) What was the relationship between the answers to a) and b) and why was this to be expected?

6. Again continue with $f(x, y) = 3x^2y$. The goal in this problem is to try to find, starting from the point $a = (1, 2)$, the direction in which $f$ grows the fastest. In light of Problem 5, we should use only unitized (normalized) directions. Compute the following. You may have to convert some of your answers to decimals to decide which answer is the biggest.

a) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(1,0)$. (easy – why?)

b) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(0,1)$.

c) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(1,1)$.

d) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(-1,-1)$. (not surprising – why?)

e) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(2,1)$.

f) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(1,2)$.

g) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(1,-1)$.

h) $\frac{\partial f}{\partial u}(1, 2)$ when $u$ is the normalized version of $(1,-4)$.

Do you think you have found the biggest? If not, look some more and report what you find.
7. On graph paper, or with carefully drawn axes, draw the vectors $(3, 2)$ and $(2, -1)$ tail to tail.
   a) Estimate the angle $\theta$ between them. (Use a protractor if you want.)
   b) Using your estimate, apply the dot product angle formula to estimate $(3, 2) \cdot (2, -1)$. (You’ll probably need a calculator.)
   c) Compute $(3, 2) \cdot (2, -1)$. Of course, this is much easier than part b), as well as exact.
   d) Use your answer to c), and the dot product angle formula, to determine $\theta$ exactly. (You’ll need a calculator with an inverse-cosine button.)

8. Suppose $g(3, -2) = 5$ and $\nabla g(3, -2) = (-4, 5)$, but that’s all you know about $g$. Estimate $g(2.8, -1.9)$.

9. Go back to Problem 6. The version of M23 for which that was written didn’t cover the dot product angle formula. So those poor students didn’t know the simple result about direction and rate of steepest ascent. (They had to guess it from the data of Problem 6.) Again using the function $f$ and the basepoint $a = (1, 2)$
   a) What is the normalized direction of steepest ascent?
   b) What is the rate of steepest ascent?
   c) Consider the contour line for $f$ through $a$. What is the slope of that contour curve at that point? *Hint:* There is a long way to do this: find an equation for that curve and compute its derivative. However, there is a much faster way given the earlier parts of this problem. You already know the slope of the line of steepest ascent at $a$. How does that relate to the slope of the contour curve?