1. Consider the equation $x^2 - x = 2x - 2$.
   a) Verify that $x = 2$ is a solution.
   b) Solve the equation.
   I am checking whether you know the difference between “verify” and “solve”.

   **Ans 1.**
   b) Plug in. On the left you get $2^2 - 2 = 2$ and on the right you get $2 \cdot 2 - 2 = 2$. Check!
   b) Here’s one way:
   \[
   x^2 - x = 2x - 2 \\
   x^2 - 3x + 2 = 0 \\
   (x - 2)(x - 1) = 0 \\
   x = 1, 2
   \]

2. Compute
   a) $(1, 2) + (3, 4)$
   b) $2(1, 2) - 3(4, -1)$
   c) $|(4, -1)|$
   d) A unit vector in the same direction as $(4, -1)$
   e) A unit vector in the opposite direction to $(4, -1)$

   **Ans 2.**
   a) $(4, 6)$
   b) $(2, 4) - (12, -3) = (-10, 7)$
   c) $\sqrt{17}$
   d) $\frac{1}{\sqrt{17}}(4, -1)$
   e) $-\frac{1}{\sqrt{17}}(4, -1) = \frac{1}{\sqrt{17}}(-4, 1)$

3. a) What is the slope of the vector $(3, -2)$? **Note:** This question uses nonstandard terminology. One doesn’t usually talk about the slope of a vector. But what we mean is the slope of any line in the plane that has this vector as a direction vector.
   b) Name two vectors with slope $-3$.

   **Ans 3.**
   a) Rise over run, or $-\frac{2}{3}$.
   b) $(1, -3)$ and $(-1, 3)$. Any vector of the form $k(1, -3)$ is correct ($k \neq 0$), and nothing else.

4. What is wrong with this sentence:
   The slopes of the lines $y = 2x + 1$ and $y = -\frac{1}{2}x + 3$ are perpendicular.

   **Ans 4.** **Lines** are perpendicular, not slopes. Slopes are numbers and numbers can’t be perpendicular. It is true that these two lines are perpendicular, because their slopes are negative reciprocals.
5. Explain why two nonzero vectors are perpendicular if and only if their dot product is 0. *Hint:* Assume and use the dot-product angle formula.

Ans 5. We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$ and $|\mathbf{u}|, |\mathbf{v}| \neq 0$. So the RHS (right-hand side) is 0 if and only if $\cos \theta = 0$, that is, if and only if $\theta = 90^\circ$, that is, if and only if the vectors are perpendicular. (Angles like 270, which also have 0 cosine, don’t arise because the angle between the tails of two arrows is always between 0 and 180.) Thus the vectors are perpendicular if and only if their dot product is 0.

Remark: We never really defined what it means for vectors to be perpendicular, since vectors are pairs of numbers, not geometric objects. But I bet you had the right idea: vectors are perpendicular if any two arrows representing them are perpendicular; in particular, if two arrows that represent them tail to tail have angle $\theta = 90^\circ$ between them.

6. Show that $(2, 3)$ and $(-6, 4)$ are perpendicular. Don’t make the problem hard; use dot products.

Ans 6. In light of Problem 5, just dot and see if you get 0: $(2, 3) \cdot (-6, 4) = -12 + 12 = 0$. Check!

7. The usual geometric interpretation of $\mathbf{u} + \mathbf{v}$ involves attaching arrows head to tail. That is, you draw any one of the arrows that represents $\mathbf{u}$, then draw the one arrow representing $\mathbf{v}$ that has its tail at the head of the first arrow. Then draw the arrow that goes from the tail of the first vector to the head of the second. That arrow represents $\mathbf{u} + \mathbf{v}$.

The “trouble” with this approach is that it is noncommutative: If you apply the same process to $\mathbf{v} + \mathbf{u}$, you get a different picture. And yet it is supposed to be true that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

So, pick any two vectors you like for $\mathbf{u}$ and $\mathbf{v}$. Pick a starting point, and create both $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$ from this point. What happens? Can you explain why it happens?

Ans 7. You get that the sum is the arrow between opposite vertices of a parallelogram, where you go around one pair of sides or the other depending on the order of your sum. It is a theorem of geometry that if you go around either way — using parallel and equal lengths on opposite sides — you do get a closed figure, a parallelogram, so the sum is the same diagonal in either order. This confirms what we already knew algebraically by adding displacements: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Everything we have done for vectors and dot products above (except slopes) works equally well for 3-tuple vectors (3 entries), or even longer vectors. The next problem illustrate this claim.

8. Compute

a) $(1, 2, 3) + 2(2, -1, 1)$

b) $|(1, -2, 3)|$

c) A unit vector in the same direction as $|(1, -2, 3)|$.

d) Show that $(1, 3, 2)$ and $(4, -2, 1)$ are perpendicular using dot products. Show enough of the computation (trivial as it is) so I can see that you know how to dot 3-tuple vectors. (This request relates to a discussion in the Quiz 2 commentary about what you have to write to communicate your mathematical knowledge.)

Ans 8. a) $(5, 0, 5)$

b) $\sqrt{1 + 4 + 9} = \sqrt{14}$
c) \( \frac{1}{\sqrt{14}}(1, -2, 3) \)
d) \( 4 - 6 + 2 = 0. \)

9. a) Find the slope-intercept equation for the line \( x = (2, 3) + t(4, -1) \).
b) Find a vector equation for the line with the cartesian equation \( 2x + 3y + 4 = 0 \).

Ans 9. a) We have a point, \((2, 3)\) and the slope, \(-1/4\). Thus \( (y - 3) = -\frac{1}{4}(x - 2) \), or \( y = -\frac{1}{4}x + \frac{7}{2} \).
b) The \( x \)-intercept is \(-2\), and for every increase of 3 in \( x \), \( y \) must be decreased by 2. Thus one vector equation is \( x = (-2, 0) + t(3, -2) \).

10. Use the dot product angle formula to find the angle between the tails of \((2, 3)\) and \((3, 1)\).

Ans 10. \( \cos \theta = \frac{(2, 3) \cdot (3, 1)}{|(2, 3)|| (3, 1)|} = \frac{9}{\sqrt{13}\sqrt{10}} \approx .789 \), and so \( \theta \approx \cos^{-1}.789 = 37.9^\circ \).

11. Suppose \( |\mathbf{u}| = 1 \) and the angle between the tails of \( \mathbf{u} \) and \((2, 1)\) is \( 60^\circ \).
a) Determine \( \mathbf{u} \cdot (2, 1) \).
b) How many different vectors \( \mathbf{u} \) are there? You can answer this by thinking geometrically; you don’t need to actually solve for these \( \mathbf{u} \)’s, and indeed, that would be difficult (though possible).

Ans 11. a) \( \mathbf{u} \cdot (2, 1) = |\mathbf{u}|| (2, 1)| \cos 60^\circ = 1 \cdot \sqrt{5} \cdot \frac{1}{2} = \sqrt{5}/2. \) That is, use the dot product angle formula, because this time the right-hand side is easy to compute.
b) Since this is a problem in the plane, there are two \( \mathbf{u} \)’s, one \( 60^\circ \) counterclockwise from \((2, 1)\) (and of unit length), the other \( 60^\circ \) clockwise. That’s all you have to say, but if you are curious, here is more. Since the angle of \((2, 1)\) is \( \tan^{-1} \frac{1}{2} \approx 26.56^\circ \), that means \( \mathbf{u} \) has angle \( 86.56^\circ \) (almost vertical) or \(-33.44^\circ \). Since \( \mathbf{u} \) has unit length, it is the vector \((\cos \alpha, \sin \alpha)\), where \( \alpha = 86.56^\circ \) or \(-33.44^\circ \). Thus the two solutions for \( \mathbf{u} \) are \((.060, .998)\) and \((.835, -.551)\). In fact, the exact formulas for \( \mathbf{u} \) are

\[
\frac{1}{10}(2\sqrt{5} - \sqrt{15}, \sqrt{5} + 2\sqrt{15}) \quad \text{and} \quad \frac{1}{10}(2\sqrt{5} + \sqrt{15}, \sqrt{5} - 2\sqrt{15})
\]

You use sine and cosine angle addition formulas to get this expressions.

12. Consider the two lines in 3-dimensional space

\[ x = (1, 2, 3) + t(1, -2, 4) \quad \text{and} \quad x = (4, \pi, e) + t(-2, 4, -8). \]

Are they parallel? How do you know?

Ans 12. Yes, they are parallel because \((-2, 4, -8) = -2(1, -2, 4)\), that is, the direction vectors of the two lines are (anti)parallel.

13. What’s wrong with the claim \( (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) \)?

Ans 13. Neither side makes any sense because you can’t dot a number (such as \( \mathbf{u} \cdot \mathbf{v} \)) with a vector (such as \( \mathbf{w} \)). So it’s not a matter of the claim being false; you can’t even ask if it is true or false because it doesn’t make sense.

3
14. Why is the following problem statement not satisfactory?

Maximize $f(x, y) = x^2y$ with the constraint $g(x, y) = x^2 + y^2$.

One says that this problem is *ill defined*.

**Ans 14.** You must set $g(x, y)$ equal to some constant, or else the set of $(x, y)$ values over which you are maximizing is not actually constrained.

15. Fill in the blank: In terms of gradients, the condition for $a$ to be a critical point of $f$ is that $\nabla f(a) = \underline{\text{.}}$.

**Ans 15.** The condition for $a$ to be a critical point of $f$ is that $\nabla f(a) = (0, 0)$. If $f$ had 3 inputs, you would fill the blank with $(0, 0, 0)$, and so on for more variables.

16. What does the condition $\nabla f(a) = \lambda \nabla g(a)$ say geometrically about the gradients $\nabla f(a)$ and $\nabla g(a)$?

**Ans 16.** At $a$, the directions of fastest growth of $f$ and $g$ (if unconstrained) are parallel, or antiparallel.

17. For this problem, let $f(x, y) = x^2y + y^2$ and let $a = (-1, 1)$.

a) Compute $\nabla f$ and $\nabla f(a)$.

b) Compare the rates of growth of $f$ from $a$ in the following directions. Be sure to unitize (the directions, not $\nabla$!) to do a fair comparison.

   i) (1, 2)

   ii) $(-4, 3)$

   iii) $(2, 3)$

   c) What is the direction of fastest growth of $f$ from $a$? (For this and the next few parts you don’t have to unitize because you are not doing comparisons of growth rates.)

   d) What is the direction of most negative growth of $f$ from $a$?

   e) What is the direction(s) of 0 growth of $f$ from $a$?

   f) If contours of $f$ were drawn with $f$-values equally spaced (e.g., contours for $f = 10, 20, 30, 40, \ldots$), in which direction from $a$ would the contour lines be closest together? Why?

   g) If one of the contour lines goes through $a$, what is the slope of that contour line at $a$?

**Ans 17.**

a) $\nabla f$ (that is, $\nabla f(x, y)$) is $(2xy, x^2+2y)$. Thus $\nabla f(a) = (-2, 3)$.

b) 

   i) The unitized direction vector is $u = (1, 2)/\sqrt{5}$ so

   $$\frac{\partial f}{\partial u}(a) = \nabla f(a) \cdot u = (-2, 3) \cdot (1, 2)/\sqrt{5} = 4/\sqrt{5} \approx 1.79.$$ 

   ii) $(-2, 3) \cdot (-4, 3)/5 = 17/5 = 3.4$

   iii) $(-2, 3) \cdot (2, 3)/\sqrt{13} = 5/\sqrt{13} \approx 1.39$

So of these directions, the one with the greatest rate of growth is $(-4, 3)$. This is not surprising, since it is almost parallel to $\nabla f(a)$ itself, which we know is always the direction of greatest rate of growth.
c) \((-2, 3) = \nabla f(a)\)

d) \(-(-2, 3) = (2, -3)\)

e) Any direction \((x, y)\) such that \((-2, 3) \cdot (x, y) = 0\), that is, any direction perpendicular to \((-2, 3)\). By trial and error, or otherwise, you can find that the solutions are \(k(3, 2)\) for any scalar \(k\). Thus there are two basic directions, \((3, 2)\) and \(-(3, 2)\), corresponding to positive and negative \(k\). Alternative Approach: Turn the problem into one about slopes and use the fact that perpendicular lines have negative reciprocal slopes.

f) The direction of fastest growth, since in this direction you rise to higher contour lines most quickly. Thus \((-2, 3)\).

g) A contour is a line of no growth, so it’s tangent is a direction of no growth. Thus part e) already told us that a tangent direction is \((3, 2)\). Using rise over run, the slope is 2/3.

18. What’s wrong with the following statements? (Each one fails to make sense, even if you knew what \(f\) was.) For each one, explain briefly.

a) The gradient of \(f\) at \((1, 2)\) is 4.

b) The rate of growth of \(f\) at \((1, 2)\) is 4.

c) The steepest ascent of \(f\) at \((1, 2)\) is \(\nabla f(1, 2)\).

d) What is the direction of fastest growth of \(f\)? (This is a question, not a statement, but what is wrong with it as a question?)

Ans 18. a) Gradients are vectors, not numbers.

b) Functions of several variables don’t have a single rate of growth, because the rate varies as you change directions from the base point. You can only talk about a numerical rate of growth for \(f\) if you specify a direction as well as a base point.

c) The what about steepest ascent? If you say the direction of steepest ascent (that is, insert the word “direction”) then the sentence makes sense. If you mean rate of steepest ascent, then the sentence makes no sense because rates are numbers, not vectors. As stated, the sentence is ambiguous, and suggests that the writer doesn’t understand the difference between directions and rates.

d) Where? In general, functions don’t have one fixed direction of fastest growth. You have to specify a base point \(a\) in order to determine a direction from there of fastest growth, namely \(\nabla f(a)\).

19. a) What is another name (or symbol) for \(\nabla f(a) \cdot u\)?

b) Why is that other name appropriate? Explain by expanding out the dot product (as a sum of products, not using cosines!) and interpreting the expanded expression.

Ans 19. a) \(\frac{df}{du}(a)\), the directional derivative of \(f\) at \(a\) in the direction \(u\).

b) \(u = (\Delta x, \Delta y)\), the amount you want to move from the basepoint \(a\). We know that

\[
f(x, y) \approx f(a) + f_x(a)\Delta x + f_y(a)\Delta y = f(a) + \nabla f(a) \cdot u.
\]

That is, the dot product captures how much the function grows in the direction \(u\) we want to go, so it is reasonable to call it a directional derivative.
20. A function \( f(x, y) \) is defined on the whole plane, but suppose we are interested in its behavior only on and inside the square shown below. Your job is to specify \( f \), by creating a few contour lines and labeling their values, so that the 6 points shown are the only critical points on and in the square and

- \( A \) and \( C \) are tied for being the global min points,
- \( D \) and \( F \) are tied for being the global max points,
- \( B \) is a local max point when only the perimeter of the square is considered, but when the interior is considered, it is actually a saddle,
- \( E \) is a local min point when only the perimeter of the square is considered, but when the interior is considered, it is actually a saddle.

All of these things have to be true simultaneously. Once you draw appropriate contour lines (this is a bit tricky) you have to explain how you know from these contour lines that you have met the requirements of the problem.

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
\end{array}
\]

**Ans 20.** Here is one way. The values on the contour lines shown below, from left to right, are 1, 2, 3, 4, 5.

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
\end{array}
\]

How do I know that \( f \) with these contours meets the requirements at points \( A \)–\( F \)? Follow the money — whoops, I mean the contours. For instance, at \( B \), if you move along the boundary, you head towards the lower contour in either direction; thus along the boundary \( B \) is a local max. However, if you move into the interior from \( B \), in most of those directions (except directions very close to the boundary) you head towards a higher contour. Thus \( B \) is a min in these directions, making \( B \) a saddle overall.