We know that the general solution to $y' = k(y - A)$ is $y(t) = A + Ce^{kt}$.

The fact is, if you know $y$ at two different times, you can estimate it at any other time, even without knowing $C$ and $k$! You do need to know $A$.

Let the difference (from $A$) be $D(t) = y(t) - A$. The key fact is that the ratio of differences is constant in the following sense: for any fixed length of time $T$,

$$\frac{D(t + T)}{D(t)}$$

is constant, independent of $t$.

**Example 1.** Suppose $T$ is the half life of some substance. That means that

$$\frac{D(T)}{D(0)} = \frac{1}{2},$$
i.e., the amount left at time $T$ is half the amount at time 0. Then for every starting time $t$,

$$\frac{D(t + T)}{D(t)} = \frac{1}{2},$$
i.e., whatever amount was left at time $t$, exactly half of that will be left at time $t + T$. (Note: for half life problems, $A = 0$; why?)

Specifically, suppose the half life of your substance is 3 hours. Then 3 hours after you start counting, half of what you started with is left. Another 3 hours later (that is, 6 hours from the start), half of that half is left, or 1/4. After 9 hours, 1/8 is left.

**Example 2.** Again consider some substance that decays to 0, that is, $A = 0$. Suppose $D(2)/D(0) = 2/3$. Then after 4 hours, $(2/3)^2 = 4/9$ is left. After 6 hours, $(2/3)^3 = 8/27 \approx .296$ is left. After 8 hours, $(2/3)^4 = 16/81 \approx .198$ is left. So after 7 hours, which is midway between 6 and 8, approximately .25 is left, because this is about midway between .296 \approx .3 and .198 \approx 2. Note that .25 is only approximate, because the decay is not linear. (In fact, exactly $(2/3)^{3.5} \approx .242$ is left.)

**Example 3.** It was 68 degrees in your house when the heater fails. Outside it is steady at 10 degrees. After 4 hours the temperature in your house has gone down to 56. What will the temperature be in 5 more hours?

**Solution.** The law of cooling is that the temperature $y(t)$ satisfies $y' = k(y - A)$. You know that $A = 10$, but you don’t know $k$. But you do know that $D(0) = 68 - 10 = 58$ and $D(4) = 56 - 10 = 46$. Thus $D(4)/D(0) = 46/58 \approx .8$ (by eyeball). Thus in 4 more hours $D$ will go down by another factor of .8, so approximately to $46 \times .8 \approx 45 \times .8 = 9 \times .4 = 36$. An extra hour maybe makes it go down by another 5% (since .8 means a decrease of 20%, and that takes 4 hours). Another 5% is another 2 degrees ($36 \times .05$), so $D$ is down to 34. That means that $y = 44$.

(To be exact, $D(9)/D(0) = (D(4)/D(0))^{9/4} = (46/58)^{9/4} \approx .594$, so $D(9) = .594 \times 58 = 34.452$ and thus $y(9) = 44.452$.)

**Proof of the Key Fact.** $D(t) = y(t) - A = (A + Ce^{kt}) - A = Ce^{kt}$. Similarly, $D(t + T) = Ce^{k(t+T)}$. Thus

$$\frac{D(t + T)}{D(t)} = \frac{Ce^{k(t+T)}}{Ce^{kt}} = \frac{e^{k(t+T)}}{e^{kt}} = e^{kT}.$$

This last expression on the right contains no $t$, so it is independent of $t$. ■

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