Example. Maximize \( P(x, y) = x^{2/3}y^{1/3} \) subject to the budget constraint \( 100x + 100y = 378,000 \), and \( x, y \geq 0 \).

Solution. The difference between this method and the original Method 2 is that now you don’t need to express the contour line in the form of one input variable as a function of the other. Rather, you find the slope of the contour line from the equation as it is given, using Implicit Differentiation. In this particular example, that means you don’t have to rework \( C = x^{2/3}y^{1/3} \) into the form \( y = C^2x^{-2} \) to find \( y' \); rather you find \( y' \) another way, as we show. This is a tremendous advantage if the contour formula is hard or impossible to solve for \( y \).

As before, we seek points \( x, y \), and a production value \( C \), such that

1) \( (x, y) \) is on the budget constraint;
2) \( (x, y) \) is on the \( C \) contour line; and
3) The budget constraint line and the \( C \) contour line are tangent at \( (x, y) \).

The budget constraint is \( x + y = 3780 \); this is a line with slope \(-1\). Thus at the point \( (x, y) \) that we are looking for, the production contour line should also have slope \(-1\). To find a formula for this slope, we differentiate \( C = x^{2/3}y^{1/3} \) as a function of \( x \), using the fact that \( y \) is a function of \( x \) on this curve even if we don’t know the formula for this function.

\[
C = x^{2/3}y^{1/3} \\
C' = \left( x^{2/3} \right)' y^{1/3} + x^{2/3} \left( y^{1/3} \right)' \\
0 = \frac{2}{3} x^{-1/3} y^{1/3} + x^{2/3} \left( \frac{1}{3} y^{-2/3} y' \right) \\
-2x^{-1/3} y^{1/3} = x^{2/3} y^{-2/3} y' \\
-2 \frac{y}{x} = y' \\
\text{(The implicit differentiation was in the middle line of this display.) Therefore} \\
\frac{-2y}{x} = -1 \quad \text{or} \quad 2y = x.
\]

Now substitute this into the budget constraint:

\[
2y + y = 3780 \\
y = 1260 \\
x = 2y = 2520.
\]

Finally,

\[
C = (2520)^{2/3}(1260)^{1/3} \approx 2000.12. \]

Note 1: The “tombstone” means that the solution (or proof) is over.

Note 2: In this problem we found the slope of the budget constraint, \(-1\), by eyeballing. In general, this constraint may also be difficult or impossible to solve for \( y \). Thus its slope \( y' \) may also need to be found by implicit differentiation.