Pictures of Functions

There are three classes of "surfaces" that can be associated with a function $f : R^n \to R^m$.

1. The **graph** of f. This is the subset of R^{n+m} defined by

$$\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} = f(\mathbf{x})\}.$$

2. The **image set** (or **parametric set**) of f,

$$f(R^n) = \{\mathbf{y} \mid \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in R^n\},$$

or more generally,

$$f(S) = \{\mathbf{y} \mid \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in S \subset R^n\}.$$

3. The **implicit set** (or **pre-image set**)

$$f^{-1}(\mathbf{0}) = \{\mathbf{x} \mid f(\mathbf{x}) = \mathbf{0}\},$$

or more generally,

$$f^{-1}(S) = \{\mathbf{x} \mid f(\mathbf{x}) \in S \subset R^m\}.$$

One also writes: Consider the set defined (implicitly) by $f(\mathbf{x}) = \mathbf{0}$, or by $f(\mathbf{x}) \in S$.

Note: Whereas one talks about *the* graph of f (the whole graph as defined above) or *the* image set (the whole image $f(R^n)$, or $f(D)$ if D is the whole domain), it doesn't usually make sense to talk about the implicit set, because one needs to know what point $\mathbf{p}$ or set S one is working back from. Whereas $\mathbf{p} = \mathbf{0}$ is the most common choice in $f^{-1}(\mathbf{p})$, it is not the only natural choice.

1. If $f : R^n \to R^m$, in what space do f's image sets lie, R^m or R^n? Where do its implicit sets lie?

2. Return to the hill associated with $f(x, y) = 40 - (x^2 + 2y^2)$ on the handout Derivatives and Gradients.

a) What sort of set is this hill: an image of f, a graph, or an implicit set?

b) Let c be a real number. Let S be the intersection of the graph of f and the plane $z = c$. What is the relationship between S and the implicit set $f^{-1}(c)$?

c) Consider the affine function $T(\mathbf{x}) = f(1, 2) + f'(1, 2)(\mathbf{x} - (1, 2))$. This is just the affine approximation to f at (1,2), so its graph should be the tangent plane to the hill at $(1, 2, f(1, 2))$. Find the graph and the image of T. Find the implicit set $T^{-1}\big(f(1, 2)\big)$. ("Find" means describe the set in some meaningful, precise way. E.g., if the set is a plane, give a formula of the sort one expects for planes.)

d) Let $c = f(1, 2) + .1$. What is the interpretation of set defined implicitly by $T(\mathbf{x}) = c$? (It's an approximation to something about the hill.)

3. **a)** For $f : R^2 \to R$ and any $r \in R$, the implicit set $f^{-1}(r)$ is called a **level curve** of f. Why?

b) If $f : R^3 \to R$ and $r \in R$, what is the implicit set $f^{-1}(r)$ called?

4. Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, and let $L(\mathbf{x}) = M\mathbf{x}$.

a) Describe the graph and image of L, and the implicit set $L^{-1}(\mathbf{0})$.

b) A flat thing that has one less dimension than the space it sits in is called a **hyperplane**. A hyperplane can always be described by an equation of the form $\mathbf{a} \cdot \mathbf{x} = b$, where $\mathbf{a}$ is a fixed vector and $\mathbf{x}$ varies over the points on the hyperplane. At least one set in part a) is a hyperplane. Find each hyperplane in part a) and figure out an equation for it of the form just described.

5. Consider the map

$$T(x, y) = (\cos x, \sin x, y).$$

a) Describe geometrically the image of T.

b) Find the implicit sets $T^{-1}(1, 0, 0)$ and $T^{-1}(\mathbf{0})$.

c) Why didn't I ask you to describe the graph of T?

6. Let $\mathbf{a} = (0, 1)$. For mapping T of Problem 5,

a) Find the differential at $\mathbf{a}$.

b) Find an algebraic description (say, one or more linear equations) for the tangent plane to the graph of T at the point $(\mathbf{a}, T(\mathbf{a}))$.

c) Find an algebraic description of the tangent plane to the *image* of T at $T(\mathbf{a})$. (Neither I nor the book has discussed how to do this, but you should be able to figure it out, at least in this case.)

7. Same as Problem 5 for $U(x, y, z) = (y \cos x, y \sin x, z)$.

8. Let

$$(u, v, w) = S(x, y) = (\cos x \sin y,\ \sin x \sin y,\ \cos y).$$

Find the image of S, and the tangent plane T to the image at $S(\pi/3, 2\pi/3)$. Can you describe T with equations using u, v, w only? Also, is there any nice restriction of the domain space for which S is 1-to-1?

9. Let $\mathbf{b} = (\mathbf{a}, f(\mathbf{a}))$ be a point on the graph G of f, where $f : R^2 \to R$, so that $\mathbf{a}$ is an ordered pair. Let $\mathbf{n} = (-\nabla f(\mathbf{a}), 1)$ and let $\mathbf{w}$ mean (x, y, z). Show that an equation of the tangent plane to G at $\mathbf{b}$ is $\mathbf{n} \cdot \mathbf{w} = \mathbf{n} \cdot \mathbf{b}$.

10. Same as Problem 5 for

$$V(x, y) = (\cos x \sin y,\ \sin x \sin y,\ \sin y).$$

Also, is there any restriction of the domain space for which V is 1-to-1?