Change of Variable in Limits

Introduction and notation. Throughout this problem set we consider the situation

$$f(a) = b, \quad f(a+h) = b+k.$$

We investigate such general questions as: Is the claim $k \to 0$ equivalent to $h \to 0$? If $y/k \to 0$, does that imply $y/h \to 0$? We investigate these for real-variable functions, but you can also ask yourself if your conclusions below work for vector functions too (with $|\mathbf{h}|$ replacing h).

1. Consider the function with domain $D = (-\infty, 0] \cup (1, \infty)$, defined by

$$f(x) = \begin{cases} x & \text{if } x \le 0 \\ x-1 & \text{if } x > 1. \end{cases}$$

a) Check that f is continuous everywhere it is defined.

b) Show that f is globally invertible. That is, there is a function $g : R \to D$ such that

$$\forall x \in D,\ g(f(x)) = x, \qquad \text{and} \qquad \forall y \in R,\ f(g(y)) = y.$$

c) Let $a = b = 0$. Show that $k \to 0$ does not imply $h \to 0$. So it's not true for all invertible functions that $k \to 0$ and $h \to 0$ are equivalent.

2. Show that even for f's without screwy domains, $k \to 0 \;\not\Longrightarrow\; h \to 0$. Hint: consider $f(x) = \sin x$ and $a = b = 0$. What goes wrong?

3. Show that if f is continuous at a, then $h \to 0 \implies k \to 0$. Show that if f^{-1} exists (globally) and f^{-1} is continuous at b, then $k \to 0 \implies h \to 0$.

4. Let f be continuously differentiable around a and suppose $f'(a) \neq 0$. Then in some neighborhood U of a,

$$mh \le k \le Mh, \tag{1}$$

where either both $m, M > 0$ or both $m, M < 0$. Note that in U, there is a one-to-one correspondence between values of h and values of k.

That (1) is true follows by the Mean Value Theorem, but we won't stop to prove it here. *Without loss of generality in what follows you may assume $m, M > 0$.*

Use ϵ's and δ's to prove carefully that, for any function G, each of the following is true:

a)

$$\lim_{h\to 0} G(h) = L \quad\iff\quad \lim_{k\to 0} G(h) = L.$$

(For a hint/reminder how to do an ϵ-δ proof for this claim, see the end of this sheet.)

b)

$$\lim_{h\to 0} \frac{G(h)}{h} = 0 \quad\iff\quad \lim_{h\to 0} \frac{G(h)}{k} = 0.$$

c)

$$\lim_{h\to 0} \frac{G(h)}{h} = 0 \quad\iff\quad \lim_{k\to 0} \frac{G(h)}{k} = 0.$$

d) What is the relevance of the earlier parts of this problem to the proof that if f is invertible around a, and $f'(a) \neq 0$, then $(f^{-1})'(b)$ exists?

5. Let u, v be variables. We say that u is *order* v, and write $u = \Theta(v)$, if there exist positive constants $m \leq M$ such that

$$m|v| \leq |u| \leq M|v|$$

for all u, v sufficiently close to 0.

a) Prove: $u = \Theta(v) \implies v = \Theta(u)$.

b) Suppose u, v, w are all related variables. Suppose $u = \Theta(v)$. Prove that:

$$\lim_{u \to 0} F(w) = L \quad \iff \quad \lim_{v \to 0} F(w) = L.$$

c) Again suppose that $u = \Theta(v)$. Prove the following are equivalent:

$$\lim_{u \to 0} F(u) = L,$$

$$\lim_{v \to 0} F(u) = L,$$

$$\lim_{u \to 0} F(v) = L.$$

Hint on Problem 4-a. To prove $[\lim_{h \to 0} G(h) = L] \implies [\lim_{k \to 0} G(h) = L]$, you are to assume you have a genie who, when you give her any ϵ, will give you a δ such that $|h| < \delta \implies |G(h) - L| < \epsilon$. Your job is to trick the genie into giving you information that allows you to name a δ' such that $|k| < \delta' \implies |G(h) - L| < \epsilon$.