Derivatives and Gradients

Note: except where a problem says find something from the definition, you can use any easy computational method that we have learned.

1. Let $f(x,y) = xy$, let $\mathbf{a} = (1,2)$, and let $\mathbf{v} = (2,3)$.

a) Find ∇f and $\nabla f(\mathbf{a})$.

b) Find $D_{\mathbf{v}} f$ and $D_{\mathbf{v}} f(\mathbf{a})$.

c) Find the rate of growth of f at $\mathbf{a}$ in a *unit* vector direction parallel to $\mathbf{v}$.

d) For what unit vector $\mathbf{u}$ is $D_{\mathbf{u}} f(\mathbf{a})$ greatest? Most negative? Zero? Why do I restrict $\mathbf{u}$ to unit vectors?

2. Show: For fixed $\mathbf{a}$ and fixed $f : R^n \to R$, the greatest value of $D_{\mathbf{u}} f(\mathbf{a})$ over all unit vectors $\mathbf{u}$ is $|\nabla f(\mathbf{a})|$.

3. Now let $f(x,y) = (xy, x^2 + y^2)$, let $\mathbf{a} = (1,2)$, and let $\mathbf{v} = (2,3)$.

a) Find $\dfrac{\partial f_1}{\partial x}$ and $\dfrac{\partial f_2}{\partial y}(\mathbf{a})$

b) Find f' and $f'(\mathbf{a})$.

c) Find $D_{\mathbf{v}} f$ and $D_{\mathbf{v}} f(\mathbf{a})$.

d) Find the rate of change of f at $\mathbf{a}$ in a *unit* vector direction parallel to $\mathbf{v}$.

e) Unlike in Problem 1d, it no longer makes sense to ask when $D_{\mathbf{u}} f(\mathbf{a})$ is greatest. Why?

4. In Edwards problem II.2.3, f has partials at $\mathbf{0}$ but isn't differentiable. It follows from a theorem that at least one partial is not continuous at $\mathbf{0}$. Find out which one(s) are not continuous. Note: except at $\mathbf{0}$, $f(x,y)$ is defined by an algebraic formula, and therefore must be (continuously) differentiable using Calc I rules. Compare the values you get near $\mathbf{0}$ using these formulas to the values for the partials at $\mathbf{0}$ that you computed when you did II.2.3.

5. Let $f(x,y) = 40 - (x^2 + 2y^2)$. Then the graph of f is an eliptical hill. A ball is put down at the point (1,2,31) on the hill. Try to show that it begins to move in the direction of steepest descent. The relevant physics principles are:

a. The force of gravity acts straight down.

b. the hill provides a supporting force on the ball perpendicular to the tangent plane of the hill at that point. It provides just enough supporting force so that the vector sum of it and gravity is tangent to the hill at that point.

c. The ball begins to move in the direction of the resultant force (since acceleration = F/m).

6. Generalize the previous problem. Show that at any point on any differentiable hill, an object placed at rest will begin to move in the direction of steepest descent.

7. Suppose f is differentiable at $\mathbf{a}$, and $\mathbf{u}$, $\mathbf{v}$ are any vectors, then $D_{\mathbf{u}} f(\mathbf{a}) + D_{\mathbf{v}} f(\mathbf{a}) = D_{\mathbf{u}+\mathbf{v}} f(\mathbf{a})$.

8. Suppose $f : R^n \to R^m$ is differentiable at $\mathbf{a}$, and define $\phi : R \to R^m$ by $\phi(t) = f(\mathbf{a} + t^2\mathbf{v})$, wherer $\mathbf{v}$ is some vector. Determine $\phi'(0)$ starting from the definition of $f'(\mathbf{a})$. (This is meant to be similar to, but a little harder than, the derivation of the formula $D_{\mathbf{v}}f(\mathbf{a}) = [f'(\mathbf{a})]\mathbf{v}$ in class.)