1. Let
$$f(x,y) = \frac{x-y}{(x+y)^3}. \tag{1}$$
We wish to find $\int_S f$ where S is the unit square $[0,1]^2$. But f is improper on S, at (0,0), so we have to use the definition of improper integral.

a) Verify that
$$f(x,y) = \frac{d}{dx}\frac{-x}{(x+y)^2} = \frac{d}{dy}\frac{y}{(x+y)^2}.$$

b) Let $A_n = [\frac{1}{n},1] \times [0,1]$. Find $\lim_{n\to\infty} \int_{A_n} f(x,y)$. (Think first about the least messy way to compute $\int_{A_n} f$.)

c) Let $B_n = [0,1] \times [\frac{1}{n},1]$. Find $\lim_{n\to\infty} \int_{B_n} f(x,y)$.

d) Let $C_n = [\frac{1}{n},1] \times [\frac{1}{n},1]$. Find $\lim_{n\to\infty} \int_{C_n} f(x,y)$.

e) Let Let $D_n = S - [0,\frac{1}{n}] \times [0,\frac{1}{n}]$. Find $\lim_{n\to\infty} \int_{D_n} f(x,y)$.

f) In light of your answers to the previous parts, $\int_S |f|$ should be ∞. Verify this. How many different approximating sequences do you need to try?

2. Continue to consider f from (1), but now let $S = [1,\infty)^2$.

a) Let $A_n = [1,n] \times [1,n]$. Find $\lim_{n\to\infty} \int_{A_n} f(x,y)$.

b) Find $\int_1^\infty \int_1^\infty f(x,y)\,dx\,dy$. Is this integral the limit of an approximating sequence as we have defined them? (If not, we are now hoping that even more ways to compute an improper integral than before will have the same value.)

c) Find $\int_1^\infty \int_1^\infty f(x,y)\,dy\,dx$.

d) Let $B_n = [1,2^n] \times [1,n]$. Find $\lim_{n\to\infty} \int_{B_n} f(x,y)$.

3. Again use f from (1), but now let $S = [1,\infty) \times [1,2] = \{(x,y) \mid x \in [1,\infty],\ y \in [1,2]\}$.

a) Before you compute the integrals below, argue conceptually that they will all be equal and finite, even though f is not nonnegative.

b) Evaluate $\int_1^\infty \int_1^2 f(x,y)\,dy\,dx$.

c) Evaluate $\int_1^2 \int_1^\infty f(x,y)\,dx\,dy$.

4. Let $g(x) = e^{-(x+y)}$, and let $R_+^2 = \{(x,y) \mid x,y \geq 0\}$. Argue for this g
$$\int_{R_+^2} g = \int_0^\infty \int_0^\infty g(x,y)\,dy\,dx.$$
I say for this g because we have not proved or even stated an improper Fubini theorem (though there is one for absolutely convergent function). You have to show that the iterated integral above must have the same value as the limit of the integrals on some valid approximating

sequence of sets. You could of course just evaluate both expressions and check that you get the same number (it happens to be 1), but I prefer an argument that does not use the actual value, but shows that one expression approaches the other (for this would give a clue how a general theorem would be proved).