

More Integration of Forms

1. Prove: If $R = [a, b] \times [c, d]$, C is the boundary of R (counterclockwise), and $P(x, y)$ is continuously differentiable, then

$$\int_R -P_y dx dy = \int_C P dx.$$

This is the second half of the proof of the rectangular Green's theorem, and proceeds essentially the same as the half I proved in class. But it helps you understand if you push the pencil too. Try to push it without looking at your notes or the proof in the text.

2. Push the pencil another way. Show that if we define

$$f(a, b) = \int_{C_y + C_x} P dx + Q dy, \quad \text{where}$$

$C_x =$ straight line from $[0, 0]$ to $[a, 0]$, $C_y =$ straight line from $[a, 0]$ to $[a, b]$,

then $f_y(x, y) = P(x, y)$. This is the second half of proving that the necessary condition $d\omega = 0$ is sufficient to make $\omega = df$ in a ball.

3. (Review of "one-point singularities") Let

$$\omega = \frac{-y dx + x dy}{x^2 + y^2}, \quad \alpha = \frac{-(y-b) dx + (x-a) dy}{(x-a)^2 + (y-b)^2}, \quad \gamma = \frac{-(y-d) dx + (x-c) dy}{(x-c)^2 + (y-d)^2},$$

Let C be the unit circle at the origin, traversed counterclockwise. Define $k = \int_C \alpha$.

- a) What is the value of k if (a, b) is inside C ? Hint: You already know that $\int_C \omega = 2\pi$ from earlier work. You also know that $d\omega = 0$ except at the origin. Finally, k equals $\int \omega$ around another circle. Why?
- b) What is k if (a, b) is outside C ?
- d) What is $\int_C \alpha + \gamma$ if both (a, b) and (c, d) are inside C ?
4. Let $F(x, y, z) = (x, y, z)$. Let S be the surface of the unit ball. Compute $\int_S \mathbf{x} \cdot N dA$ using the divergence theorem. Note: N is the unit normal, and on the *unit* ball, so is \mathbf{x} . Thus $\mathbf{x} \cdot N = 1$ everywhere on S .

5. Let H be the upper unit hemisphere, so that the unit circle C on the xy plane (viewed as part of R^3) is its boundary. C is also the boundary of the unit disk D in the xy plane. Use Stokes' Theorem to turn

$$\int_C y dx + z dy + x dz$$

- a) into an integral on H ,
- b) into an integral on D . Evaluate this last integral (all the stuff that we haven't learned how to evaluate goes away).
6. Suppose $f: R^n \rightarrow R$. Define $\nabla^2 f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$. Show that if $\omega = df$, then $\text{div } w = \nabla^2 f$.