More Contour Integration and Exact Differentials

1. I stated in class that if $\omega = df$, then line integrals of ω are independent of path. I showed that $\omega = df$ is true if ω is the 1-form for gravity. But I also showed that if $\omega = (-y\,dx + x\,dy)/(x^2+y^2)$, then $\omega = d\theta$, where $\theta = \arctan(y/x)$. Therefore, $\int_C \omega$ should be 0 for this ω, if C is a closed path. Yet I claimed that, for any circle C around the origin, $\int_C \omega = 2\pi$. Something doesn't jibe. What's the explanation?

2. Consider the two circles

$$C_1 : x^2 + y^2 = 1, \qquad\qquad C_1 : x^2 + (y+2)^2 = 16.$$

Let D be the region between the circles. In the integrals below, go around each circle counterclockwise.

a) Draw C_1, C_2 and D.

b) Consider $\omega = Pdx + Qdy = x\,dx + y\,dy$. Verify that $Q_x - P_y = 0$ on D. Therefore the two integrals

$$\int_{C_1} \omega \qquad \text{and} \qquad \int_{C_2} \omega \tag{2}$$

must be equal. Verify this by direct calculation. Which integral was easier to compute?

c) Draw a picture of C_1 with the vectors of $F = (P, Q)$ attached. That is, at point $\mathbf{x}$, attach $F(\mathbf{x})$ with its tail at $\mathbf{x}$. Explain why it is now obvious that $\int_{C_1} \omega = 0$. However, it is not obvious (to me) from the same sort of picture that $\int_{C_2} \omega = 0$.

d) For this ω, $Q_x - P_y = 0$ inside C_1 as well as in D. Why does this ensure that both integrals in (2) are 0?

e) Now consider

$$\alpha = \frac{-y\,dx + x\,dy}{x^2+y^2}.$$

Verify that $d\alpha = 0$ in D. Therefore

$$\int_{C_1} \alpha \;=\; \int_{C_2} \alpha. \tag{3}$$

Verify this by direct computation, or at least try. Which integral is easier to compute? (This part of the problem is supposed to amaze you – that two integrals that look so different and are not related by the Chain Rule must be equal.) Note that $d\alpha$ isn't 0 everywhere inside of C_1, because $d\alpha$ is not defined everywhere inside C_1. Where isn't it defined? Thus no theorem ensures that $\int_{C_1} \alpha = 0$. And sure enough, $\int_{C_1} \alpha \neq 0$.

3. Consider

$$\omega = e^{(x+2y)^2} dx + 2e^{(x+2y)^2} dy.$$

Show that $\omega = df$ for some f. (Whereas for some of the similar problems in Edwards you can actually integrate to find f instead of using the necessary derivative condition, here you cannot integrate, at least not in closed form.)

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