Completing the Square, Gaussian Elimination, and Quadratic Forms

In high school you learned how to complete the square:

$$x^2 + 6x = x^2 + 6x + 9 - 9 = (x+3)^2 - 3^2.$$

You could have done this in more complicated situations:

$$x^2 + 6xy = x^2 + 6xy + 9y^2 - 9y^2 = (x+3y)^2 - 9y^2,$$

or even

$$2x^2 + 8xy + 4y = 2(x^2 + 4xy + 4y^2) - 8y^2 + 4y = 2(x+2y)^2 - 8(y^2 + y/2),$$

and now you could go on to complete the square for $(y^2 + y/2)$, obtaining $(y + \frac{1}{4})^2 - \frac{1}{16}$. Note that completing the first square got all the x-terms out of the way.

1. Consider the quadratic form

$$Q(x, y, z) = x^2 + 2xy + 3y^2 - 4xz. \qquad (1)$$

a) Complete the square and thus express $Q(x, y, z)$ in the form $aX^2 + bY^2 + cZ^2$, where a, b, c are numbers and X, Y, Z are expressions in x, y, z. (Actually, Y involves only y and z. How about Z?).

b) Is Q positive definite, negative definite, or neither? How do you know based on part a)?

2. Find a symmetric matrix A so that $Q(\mathbf{x})$ of (1) equals $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$.

3. Suppose L must be a lowertriangular square matrix with all 1s on the main diagonal, U must be an uppertriangular square matrix with all 1s on the main diagonal, and D must be diagonal. It is a fact that for any square matrix A, there is *at most* one factorization of the form

$$A = LDU.$$

Such a factorization is called an LDU-factorization. (Optional: prove this uniqueness.)

a) Suppose A is symmetric. How are L and U related? Hint: $(LDU)^{\mathrm{T}}$ is another LDU factorization of A.

b) Find the LDU-factorization for A of Problem 2. (This is easy if your linear algebra course covered how Gaussian elimination automatically provides an LDU-factorization.)

c) Use your factorization in b) to group $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ as $(\mathbf{x}^{\mathrm{T}}L)D(U\mathbf{x})$. Notice anything interesting relative to problem 1?

4. Let A_i be the upper left $i \times i$ corner of A.

a) If LDU is an LDU-factorization of A, show that $A_i = L_i D_i U_i$ for each i.

b) Express $\det(A_i)$ easily in terms of something from LDU.

c) Prove the principal determinant tests for positive definite and negative definite quadratic forms (at least for matrices that have LDU factorizations).

5. Prove that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does *not* have any LDU factorization.

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